

**At the Edge of Criticality:  
Markov Chains with Asymptotically Zero Drift**

*Denis Denisov<sup>1</sup>, Dmitry Korshunov<sup>2</sup> and Vitali Wachtel<sup>3</sup>*

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<sup>1</sup>The University of Manchester, UK. E-mail: denis.denisov@manchester.ac.uk

<sup>2</sup>Lancaster University, UK. E-mail: d.korshunov@lancaster.ac.uk

<sup>3</sup>Augsburg University, Germany E-mail: vitali.wachtel@math.uni-augsburg.de



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# Chapter 1

## Overview

Let  $X = \{X_n, n \geq 0\}$  be a time homogeneous Markov chain taking values in  $\mathbb{R}^+$ . Denote by  $\xi(x)$ ,  $x \in \mathbb{R}^+$ , a random variable corresponding to the jump of the chain at point  $x$ , that is, a random variable with distribution

$$\begin{aligned}\mathbb{P}\{\xi(x) \in B\} &= \mathbb{P}\{X_{n+1} - X_n \in B \mid X_n = x\} \\ &= \mathbb{P}_x\{X_1 \in x + B\}, \quad B \in \mathcal{B}(\mathbb{R});\end{aligned}$$

hereinafter the subscript  $x$  denotes the initial position of the Markov chain  $X$ , that is,  $X_0 = x$ .

Denote the  $k$ th moment of the jump at point  $x$  by  $m_k(x) := \mathbb{E}\xi^k(x)$ . We say that a Markov chain has *asymptotically zero drift* if  $m_1(x) = \mathbb{E}\xi(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

The study of processes with asymptotically zero drift was initiated by Lamperti in a series of papers [48, 49, 50].

In [48] he has shown that if  $\limsup X_n = \infty$  and  $\mathbb{E}|\xi(x)|^{2+\delta}$  are bounded for some positive  $\delta$  then

- $2xm_1(x) \leq m_2(x) + O(x^{-\delta})$  yields recurrence of  $X_n$ ,
- $2xm_1(x) \geq (1 + \varepsilon)m_2(x)$  yields transience of  $X_n$ .

In [50] he has proven that  $2xm_1(x) + m_2(x) \leq -\varepsilon$  is sufficient for the positive recurrence of  $X_n$  and that  $2xm_1(x) + m_2(x) \geq \varepsilon$  implies that  $X_n$  is non-positive (either null-recurrent or transient). These criteria have been improved later by Menshikov, Asymont and Yasnogorodskii [51]. Instead of existence of  $2 + \delta$  bounded moments they assume that  $\mathbb{E}\xi^2(x) \log^{2+\delta}(1 + |\xi(x)|)$  are bounded. Moreover, they have determined more precise boundaries between positive recurrence, null-recurrence and transience.

One of the most popular examples of a Markov chain with asymptotically zero drift is a driftless random walk conditioned to stay positive. This process is an  $h$ -transform of a random walk killed at leaving  $\mathbb{R}^+$ . If the original random walk has finite second moments then the transformed process has drift of order  $1/x$ , that is,  $xm_1(x) \rightarrow c_1 > 0$ . But the second moment of the transformed process is finite if and only if the third moment of the original walk is finite, see calculations in Section 7.1. Therefore, Lamperti's criterion on transience is not always applicable to this chain.

This was our motivation to look for appropriate conditions for transience, null-recurrence and positive recurrence in terms of truncated moments and tail probabilities of jumps  $\xi(x)$ . For any  $s > 0$  we denote  $s$ -threshold of the  $k$ th moment of jump at the state  $x$  by

$$m_k^{[s]}(x) := \mathbb{E}\{\xi^k(x); |\xi(x)| \leq s\}.$$

In Chapter 2 we introduce a classification of Markov chains with asymptotically zero drift, which relies on relations between  $m_1^{[s(x)]}$  and  $m_2^{[s(x)]}$ . Additional assumptions are expressed in terms of truncated moments of higher order and tail probabilities of jumps. Another, more important, contrast to previous results on recurrence/transience is the fact that we do not use concrete Lyapunov test functions (like  $x^2$ ,  $\log^a x$  or  $x^2 \log x \log \log x$ ). Instead, we try to construct an abstract Lyapunov function which looks similar to functions which characterise the behaviour of diffusions with drift  $m_1(x)$  and diffusion coefficient  $m_2(x)$ .

Chapter 3 is devoted to the limiting behaviour of transient chains. Here we prove that if the drift  $m_1^{[s(x)]}(x)$  behaves like  $\mu/x$  and the second moment  $m_2^{[s(x)]}(x)$  converges to a positive constant  $b$  then  $X_n^2/n$  converges to a  $\Gamma$ -distribution with parameters depending on  $\mu$  and  $b$ . This result generalises papers by Lamperti [49], Kersting [40] and Klebaner [42], where convergence towards  $\Gamma$  were proved under more restrictive assumptions. For processes converging to  $\Gamma$ -distribution we determine also the asymptotic behaviour of the cumulative renewal function. For processes satisfying  $xm_1^{[s(x)]}(x) \rightarrow \infty$  we prove weak and strong laws of large numbers, global and integro-local versions of CLT. A global CLT is just a slight improvement of the recent result by Menshikov and Wade [55]. As a sequence of an integro-local CLT we obtain also an integro-local renewal theorem.

In Chapter 4 we introduce a general strategy of change of measure for Markov chains with asymptotically zero drift. This is the most important ingredient in our approach to recurrent chains. There are two different approaches known in the literature. The first one is based on the use of Lyapunov functions, which lead to certain super- or submartingales. Usually one takes a combination of certain elementary functions. Such test functions are easy to analyse and they lead quite often to qualitatively optimal estimates for original recurrent Markov chains. Some examples can be found in [6, 34, 52]. Many further examples of using this approach are presented in [53]. The main disadvantage of this approach is the fact that it is very hard to derive exact asymptotics from such test functions. In [17] we have proposed an alternative method, which is based on the construction of a positive harmonic function  $V(x)$  for  $X_n$  killed at entering a fixed compact set  $[0, \hat{x}]$ . This means that  $V$  is a positive solution to

$$V(x) = \mathbb{E}\{V(x + \xi(x)); x + \xi(x) > \hat{x}\}, \quad x \geq 0. \quad (1.1)$$

This implies that  $V(X_n)\mathbb{I}\{\min_{1 \leq k \leq n} X_k > \hat{x}\}$  is martingale and that we may perform the Doob  $h$ -transform with this function. In order to analyse the transformed Markov chain we need to know the asymptotic properties of the function  $V$ . It



follows from the construction that this function is regularly varying with known index, but this information is not sufficient. And it is quite difficult to derive further properties of  $V$ . To overcome this obstacle we have imposed extra moment restrictions on the jumps  $\xi(x)$  and have derived the exact asymptotics for the tail of the stationary distribution for positive recurrent chains with the drift proportional to  $1/x$ . This example shows that the harmonic function is useful in deriving asymptotics, but this function is hard to analyse. This is due to the fact this function is implicitly defined, as a solution to (1.1). In Chapter 4 we present a combination of two approaches described above. More precisely, we construct a sufficiently smooth Lyapunov function  $U_p$  such that

$$-C \frac{m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} p(x) \leq \frac{\mathbb{E} U_p(x + \xi(x))}{U_p(x)} < 0, \quad (1.2)$$

where  $p$  is an integrable function. This function is almost harmonic and has explicit form. From the second inequality in (1.2) we see that the Doob transform with this function leads to a substochastic kernel. But, due to the first inequality, we have a good control over the loss of mass. The smoothness of  $U_p$  allows to determine the asymptotic behaviour of the moments of the chain embedded in  $Q$  and, consequently, we may apply our results from Chapter 3.

Chapter 5 is devoted to the study of the limiting behaviour of recurrent chains with the drift proportional to  $1/x$ . We derive asymptotics for a stationary measure  $\pi$ . We also show that  $\mathbb{P}\{X_n > x\} \sim F(n/x^2)\pi(x, \infty)$  for positive recurrent chains. Finally we determine the tail behaviour of recurrence times and prove a limit theorem for  $X_n$  conditioned on the event that the chain does not come back to a compact  $[0, \hat{x}]$ . All these asymptotics are of power type and are determined by the behaviour of the ratio  $2 \frac{m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)}$ . The situation changes in the case when the drift goes to

zero slower than  $1/x$ . In Chapter 6 we show that if  $m_1^{[s(x)]}(x)$  is of order  $x^{-\beta}$  then  $m_k^{[s(x)]}(x)$  with  $k \leq [1/\beta] + 1$  are important for the behaviour of stationary distributions and pre-limiting tails. Here we obtain Weibull-like asymptotics.

In Chapter 7 we consider some possible applications of our results. Processes with asymptotically zero drift naturally appear in various stochastic models like random billiards, see Menshikov et al. [54], and random polymers, see Alexander [4], Alexander and Zygouras [5], De Coninck et al. [15]). Probably the largest class, where such chains appear, are critical and near-critical branching processes. In critical branching processes one has typically a linearly growing second moments of jumps, but considering the square root of the process one gets bounded second moments and decreasing to zero drift. Then we can apply our theorems to this transformation. As a result we get limit theorems for population size-dependent processes with migration of particles. As far as we know, there are no papers in the literature, where a combination of size dependence and migration has been considered. We've found also out that processes with asymptotically zero drift can be used in the study of risk processes with reserve-dependent premium rate. More precisely, we have derived upper and lower bounds for ruin probabilities in the case

when the premium rate approaches from above the critical value for the model with constant rate. Besides these two main examples we consider also random walks conditioned to stay positive and reflected random walks.

In the last chapter we consider asymptotically homogeneous Markov chains, that is,  $\xi(x) \Rightarrow \xi$  as  $x \rightarrow \infty$ . This means that at large distances from zero one can approximate  $X_n$  by a random walk. Similar to Chapters 5 and 6 we study the asymptotic behaviour of the stationary distribution and pre-stationary distributions of  $X_n$  in the case when the limiting variable  $\xi$  has negative mean and satisfies the Cramer condition. It turns out that the behaviour of these distributions depends on the rate of convergence of  $\xi(x)$  to  $\xi$ .

## Chapter 2

# Lyapunov functions for classification of Markov chains

### 2.1 Heuristics coming from diffusion processes

Consider a diffusion on  $\mathbb{R}^+$  with drift  $m_1(x)$  and diffusion coefficient  $m_2(x)$ . In the case of stable diffusion, the invariant density function  $p(x)$  solves the Kolmogorov forward equation

$$0 = -\frac{d}{dx}(m_1(x)p(x)) + \frac{1}{2}\frac{d^2}{dx^2}(m_2(x)p(x)),$$

which has the following solution:

$$p(x) = \frac{2c}{m_2(x)} e^{\int_0^x \frac{2m_1(y)}{m_2(y)} dy}, \quad c > 0. \quad (2.1)$$

It follows that a diffusion process possesses a probabilistic invariant distribution—is positive recurrent—if and only if

$$\text{the function } \frac{1}{m_2(x)} e^{\int_0^x \frac{2m_1(y)}{m_2(y)} dy} \text{ is integrable.} \quad (2.2)$$

It is also known that a diffusion process is recurrent if

$$\text{the function } e^{-\int_0^x \frac{2m_1(y)}{m_2(y)} dy} \text{ is not integrable at infinity;} \quad (2.3)$$

see, e.g. [35, Theorem 7.3]; and vice versa, it is transient if

$$\text{the function } e^{-\int_0^x \frac{2m_1(y)}{m_2(y)} dy} \text{ is integrable at infinity.} \quad (2.4)$$

As one can see, the classification of diffusions relies on the asymptotic behaviour of the ratio  $2m_1(x)/m_2(x)$ . In this chapter we are going to introduce criteria for transience, recurrence and positive recurrence of discrete time Markov chains by constructing Lyapunov functions which depend on the ratio of truncated moments of the chain in a very similar to (2.2)–(2.4) way.

In this chapter,  $r(x) > 0$  is always a decreasing function which is non-integrable at infinity, that is,

$$R(x) := \int_0^x r(y)dy \rightarrow \infty \text{ as } x \rightarrow \infty. \quad (2.5)$$

The function  $R(x)$  is concave because  $r(x)$  is decreasing.

If  $r(x) > 0$  is additionally differentiable and, for some  $c > 0$ ,

$$0 \geq r'(x) \geq -cr^2(x) \text{ for all sufficiently large } x, \quad (2.6)$$

then, for all  $h > 0$  eventually in  $x$ ,

$$\begin{aligned} \frac{1}{r(x)} - \frac{1}{r(x+h/r(x))} &= \int_x^{x+h/r(x)} \frac{r'(y)}{r^2(y)} dy \\ &\geq -c \int_x^{x+h/r(x)} dy = -c \frac{h}{r(x)}. \end{aligned}$$

Therefore,

$$r(x+h/r(x)) \geq \frac{r(x)}{1+ch}, \quad h > 0. \quad (2.7)$$

Similarly,

$$r(x-h/r(x)) \leq \frac{r(x)}{1-ch}, \quad h \in (0, 1/c). \quad (2.8)$$

The lower bound (2.7) implies that, for any  $h > 0$ ,

$$\begin{aligned} R(x+h/r(x)) &= R(x) + \int_x^{x+h/r(x)} r(y)dy \\ &\geq R(x) + \frac{h}{r(x)} r(x+h/r(x)) \\ &\geq R(x) + \frac{h}{1+ch}. \end{aligned} \quad (2.9)$$

On the other hand, for any  $h > 0$ ,

$$\begin{aligned} R(x+h/r(x)) &= R(x) + \int_0^{h/r(x)} r(x+y)dy \\ &\leq R(x) + \frac{h}{r(x)} r(x) \leq R(x) + h, \end{aligned} \quad (2.10)$$

so hence

$$R(x) + \frac{h}{1+ch} \leq R(x+h/r(x)) \leq R(x) + h, \quad (2.11)$$

Similarly,

$$R(x) - \frac{h}{1 - ch} \leq R(x - h/r(x)) \leq R(x) - h, \quad (2.12)$$

where the first inequality is valid for  $h \in (0, 1/c)$ . So,  $1/r(x)$  is a natural step size responsible for constant increase of the function  $R(x)$ . Moreover, (2.11) and (2.12) imply that

$$R(x + o(1/r(x))) = R(x) + o(1) \quad \text{as } x \rightarrow \infty. \quad (2.13)$$

## 2.2 Positive recurrence

### 2.2.1 Positive recurrence in the case of bounded second moments

Let  $x_0$  be such that

$$\frac{2m_1^{[x]}(x)}{m_2^{[x]}(x)} \leq -r(x) \quad \text{for all } x > x_0; \quad (2.14)$$

in general, this means that the drift to the left dominates the diffusion and the corresponding Markov chain  $X_n$  is typically positive recurrent.

In the next theorem we show that—similarly to diffusion processes—the chain  $X_n$  is positive recurrent provided that the function

$$\frac{1}{b(x)} e^{-R(x)} = \frac{1}{b(x)} e^{-\int_0^x r(y) dy} \quad \text{is integrable,} \quad (2.15)$$

where  $b(x) > 0$  is a differentiable function such that

$$\liminf_{x \rightarrow \infty} \frac{m_2^{[x]}(x)}{b(x)} > 0. \quad (2.16)$$

under some additional technical conditions on  $r(x)$  and on the function

$$W(x) := e^{R(x)} \int_x^\infty \frac{1}{b(y)} e^{-R(y)} dy,$$

which is a well defined function due to (2.15).

The standard approach to prove positive recurrence is to construct a positive test function  $L(x)$  such that, for some  $x_*$  and  $\varepsilon > 0$ ,

$$\mathbb{E}\{L(X_1) - L(x) \mid X_0 = x\} \leq -\varepsilon \quad \text{for all } x > x_*. \quad (2.17)$$

In the next theorem sufficient conditions are given that guarantee that the test function

$$L(x) := \int_0^x W(y) dy, \quad x > 0, \quad (2.18)$$

and  $L(x) = 0$  on  $\mathbb{R}^-$ , is appropriate. In particular, it agrees with the case  $r(x) \equiv \varepsilon > 0$  where the most natural choice of test function is linear one; and with the case  $r(x) = \mu/x$  where the most effective test function is  $x^2$ .

**Theorem 2.1.** *Let the conditions (2.14), (2.15) and (2.16) hold for some decreasing and differentiable function  $r(x)$ , so  $W(x)$  is twice differentiable. Let  $W(x)$  be convex increasing and, for some constants  $c_1$  and  $c_2$ ,*

$$W'(x) \leq c_1 W(x)/x, \quad (2.19)$$

$$W''(x) \leq c_2 W(x)/x^2 \quad \text{for all } x > 0. \quad (2.20)$$

Let

$$\mathbb{E}\{\xi^3(x), \xi(x) \in [0, x]\} = o(x^2/W(x)), \quad (2.21)$$

$$\mathbb{E}\{\xi(x)W(\xi(x)), \xi(x) > x\} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (2.22)$$

Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is positive recurrent.

The conditions (2.21) and (2.22) are fulfilled if, for example,

$$\text{the family } \{\xi^+(x)W(\xi^+(x)), x \geq 0\} \text{ is uniformly integrable.} \quad (2.23)$$

**Corollary 2.2.** *Let, for some  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq x\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}} \leq -\frac{1+\varepsilon}{x} \quad \text{for all } x > x_0.$$

Let the truncated second moments  $\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}$  be bounded away from zero,

$$\mathbb{E}\{\xi^3(x), \xi(x) \in [0, x]\} = o(x), \quad (2.24)$$

$$\mathbb{E}\{\xi^2(x), \xi(x) > x\} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (2.25)$$

Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is positive recurrent.

Notice that both (2.24) and (2.25) hold provided that the family of random variables  $\{(\xi^+(x))^2, x > 0\}$  is uniformly integrable.

*Proof of Corollary 2.2.* follows if we take  $r(x) = (1 + \varepsilon/2)/x$  and  $b(x) = 1$ , then  $R(x) = (1 + \varepsilon/2) \log x$ ,  $e^{-R(x)} = 1/x^{1+\varepsilon/2}$ ,  $W(x) = 2x/\varepsilon$ , and  $L(x) = x^2/\varepsilon$ .  $\square$

Notice that the last corollary where a quadratic Lyapunov function appears provides too restrictive assumptions on jumps, while the classical Lamperti's criterion says that if only  $2xm_1(x) + m_2(x) \leq -\varepsilon$  holds for all  $x > x_0$ , then the set  $(-\infty, x_0]$  is positive recurrent.

Let  $\log_{(m)} x$  denote the  $m$ th iteration of the logarithm of  $x$ .

**Corollary 2.3.** *Let, for some  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq x\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}} \leq -\frac{1}{x} - \frac{1}{x \log x} - \dots - \frac{1}{x \log x \dots \log_{(m-1)} x} - \frac{1+\varepsilon}{x \log x \dots \log_{(m)} x}$$

for all sufficiently large  $x$ . Let the truncated second moments  $\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}$  be bounded away from zero,

$$\mathbb{E}\{\xi(x)^3; \xi(x) \in [0, x]\} = o\left(\frac{x}{\log x \dots \log_{(m)} x}\right), \quad (2.26)$$

and

$$\mathbb{E}\{(\xi^+(x))^2 \log \xi^+(x) \dots \log_{(m)} \xi^+(x); \xi(x) > x\} \rightarrow 0. \quad (2.27)$$

Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is positive recurrent.

This result was first established by Menshikov et al. [51] under the condition that moments of order  $x^2 \log^{2+\delta} x$  are bounded. Notice that both (2.26) and (2.27) hold provided that the family

$$\{(\xi^+(x))^2 \log \xi^+(x) \dots \log_{(m)} \xi^+(x), x > 0\} \text{ is uniformly integrable.}$$

*Proof of Corollary 2.3.* Consider

$$r(x) := \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m-1)} x} + \frac{1 + \varepsilon/2}{x \log x \dots \log_{(m)} x}$$

and  $b(x) = 1$ ; then

$$\begin{aligned} R(x) &= \log x + \log \log x + \dots + \log_{(m)} x + (1 + \varepsilon/2) \log_{(m+1)} x, \\ e^{-R(x)} &= \frac{1}{x \cdot \log x \cdot \dots \cdot \log_{(m-1)} x \cdot \log_{(m)}^{1+\varepsilon/2} x}, \\ W(x) &= \frac{2}{\varepsilon} x \log x \cdot \dots \cdot \log_{(m-1)} x \cdot \log_{(m)} x, \\ L(x) &\sim \frac{1}{\varepsilon} x^2 \log x \cdot \dots \cdot \log_{(m-1)} x \cdot \log_{(m)} x. \end{aligned}$$

□

The next corollary concerns the case where the second moment of jumps is vanishing at infinity.

**Corollary 2.4.** *Let, for some  $\alpha > 0$ ,  $c_1, c_2 > 0$ , and  $x_0 > 0$ ,*

$$\begin{aligned} \mathbb{E}\{\xi(x); |\xi(x)| \leq x\} &\leq -c_1/x^{1+\alpha} \quad \text{for all } x > x_0, \\ \mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\} &\sim c_2/x^\alpha \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Let

$$\mathbb{E}\{\xi^3(x), \xi(x) \in [0, x]\} = o(x^{1-\alpha}), \quad (2.28)$$

$$\mathbb{E}\{\xi^{2+\alpha}(x), \xi(x) > x\} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (2.29)$$

If  $2c_1/c_2 > 1 + \alpha$ , then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is positive recurrent.

In the case  $\alpha \in (0, 1)$ , both (2.28) and (2.29) hold provided that the family of random variables  $\{(\xi^+(x))^{2+\alpha}, x > 0\}$  is uniformly integrable.

*Proof of Corollary 2.4.* follows if we take  $c \in (1 + \alpha, 2c_1/c_2)$ ,  $r(x) = c/x$  and  $b(x) = 1/x^\alpha$ , then  $R(x) = c \log x$ ,  $e^{-R(x)} = 1/x^c$ ,

$$W(x) = x^c \int_x^\infty \frac{y^\alpha}{y^c} dy = \frac{x^{\alpha+1}}{\alpha - c + 1},$$

and  $L(x) = x^{2+\alpha}/(\alpha - c + 1)(2 + \alpha)$ .  $\square$

The advantage of Theorem 2.1 is that it covers all these functions in a unified way; the basic condition (2.15) is motivated by the existence condition (2.2) for stationary density of a diffusion process. But at the same time this link to diffusion processes results in necessity of finite second moments which is natural in Corollaries 2.2 and 2.3 while there are other examples where the existence of second moments of jumps is clearly excessive. In the next subsection we discuss amended moment conditions for drifts like  $-1/x^\alpha$ ,  $0 < \alpha < 1$ , which may be characterised by the convergence  $x\mu_1(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

*Proof.* We consider the test function (2.18) for which we need to show (2.17). By the construction,  $L'(x) = W(x)$  and

$$L''(x) = W'(x) = r(x)W(x) - 1/b(x). \quad (2.30)$$

Since  $W(x)$  is increasing,

$$L(x) \leq xW(x) \quad \text{for all } x. \quad (2.31)$$

Fix  $\varepsilon > 0$ . Then it follows from (2.19) and (2.20) that, by Taylor's theorem

$$\begin{aligned} W(x + \varepsilon x) &= W(x) + W'(x)\varepsilon x + \frac{1}{2}W''(x + \theta\varepsilon x)\varepsilon^2 x^2 \\ &\leq W(x) + c_1\varepsilon W(x) + \frac{c_2\varepsilon^2}{2}W(x + \theta\varepsilon x) \\ &\leq W(x) + c_1\varepsilon W(x) + \frac{c_2\varepsilon^2}{2}W(x + \varepsilon x), \end{aligned}$$

because  $W$  is increasing. Take  $\varepsilon = 1/\sqrt{c_2}$ . Then

$$W(x + \varepsilon x) \leq 2(1 + c_1\varepsilon)W(x).$$

Therefore, there is a  $c_3 < \infty$  such that

$$W(2x) \leq c_3W(x) \quad \text{for all } x > 0. \quad (2.32)$$

Let us prove that the mean drift of  $L(x)$  is negative and bounded away from zero for all sufficiently large  $x$ . First we analyse Taylor's theorem for the function  $L$ , with the Lagrange form of the remainder, here  $x, x + y > 0$ :

$$\begin{aligned} L(x + y) - L(x) &= L'(x)y + L''(x)\frac{y^2}{2} + L'''(x + \theta y)\frac{y^3}{6} \\ &= W(x)y + r(x)W(x)\frac{y^2}{2} - \frac{y^2}{2b(x)} + W''(x + \theta y)\frac{y^3}{6}, \end{aligned} \quad (2.33)$$



where  $0 \leq \theta = \theta(x, y) \leq 1$ . Since  $W(x)$  is assumed to be convex,  $W'' \geq 0$  and hence

$$L(x+y) - L(x) \leq W(x)y + r(x)W(x)\frac{y^2}{2} - \frac{y^2}{2b(x)} \text{ for all } y \in [-x, 0]. \quad (2.34)$$

Next, by the condition (2.20), for  $y \in [-x/2, x]$ ,

$$\begin{aligned} W''(x+\theta y) &\leq 4c_2 \frac{W(x+\theta y)}{x^2} \\ &\leq 4c_2 \frac{W(2x)}{x^2} \leq 4c_2 c_3 \frac{W(x)}{x^2}, \end{aligned} \quad (2.35)$$

because the function  $W(x)$  increases and  $W(2x) \leq c_3 W(x)$ . Substituting this into (2.33) we get

$$L(x+y) - L(x) \leq W(x)y + r(x)W(x)\frac{y^2}{2} - \frac{y^2}{2b(x)} + c_4 \frac{W(x)}{x^2} y^3 \text{ for all } y \in [0, x]. \quad (2.36)$$

Applying the fact that  $L$  is increasing and the inequalities (2.31) and (2.32), we deduce that

$$L(x+y) \leq L(2y) \leq 2yW(2y) \leq 2c_3 yW(y) \text{ for all } y > x. \quad (2.37)$$

Now we are ready to estimate the mean drift of  $L(X_n)$ . We start with the following upper bound

$$\begin{aligned} \mathbb{E}L(x+\xi(x)) - L(x) &\leq \mathbb{E}\{L(x+\xi(x)) - L(x); \xi(x) \in [-x, 0]\} \\ &\quad + \mathbb{E}\{L(x+\xi(x)) - L(x); \xi(x) \in [0, x]\} \\ &\quad + \mathbb{E}\{L(x+\xi(x)); \xi(x) > x\}. \end{aligned} \quad (2.38)$$

It follows from (2.34) that

$$\begin{aligned} &\mathbb{E}\{L(x+\xi(x)) - L(x); \xi(x) \in [-x, 0]\} \\ &\leq W(x)\mathbb{E}\{\xi(x); \xi(x) \in [-x, 0]\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); \xi(x) \in [-x, 0]\} \\ &\quad - \frac{1}{2b(x)}\mathbb{E}\{\xi^2(x); \xi(x) \in [-x, 0]\}. \end{aligned} \quad (2.39)$$

It follows from (2.36) that

$$\begin{aligned} &\mathbb{E}\{L(x+\xi(x)) - L(x); \xi(x) \in [0, x]\} \\ &\leq W(x)\mathbb{E}\{\xi(x); \xi(x) \in [0, x]\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); \xi(x) \in [0, x]\} \\ &\quad - \frac{1}{2b(x)}\mathbb{E}\{\xi^2(x); \xi(x) \in [0, x]\} + c_4 \frac{W(x)}{x^2}\mathbb{E}\{\xi^3(x); \xi(x) \in [0, x]\} \\ &\leq W(x)\mathbb{E}\{\xi(x); \xi(x) \in [0, x]\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); \xi(x) \in [0, x]\} \\ &\quad - \frac{1}{2b(x)}\mathbb{E}\{\xi^2(x); \xi(x) \in [0, x]\} + o(1) \text{ as } x \rightarrow \infty, \end{aligned} \quad (2.40)$$

due to the condition (2.21). Finally, it follows from (2.37) by the condition (2.22) that

$$\begin{aligned} \mathbb{E}\{L(x + \xi(x)); \xi(x) > x\} &\leq 2c_3 \mathbb{E}\{\xi(x)W(\xi(x)); \xi(x) > x\} \\ &\rightarrow 0 \quad \text{as } x \rightarrow \infty. \end{aligned} \quad (2.41)$$

Substituting the upper bounds (2.39)–(2.41) into (2.38) we deduce that

$$\begin{aligned} &\mathbb{E}\{L(x + \xi(x)) - L(x)\} \\ &\leq W(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\} \\ &\quad - \frac{1}{2b(x)}\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\} + o(1) \\ &= W(x)\frac{m_2^{[x]}(x)}{2}\left(\frac{2m_1^{[x]}(x)}{m_2^{[x]}(x)} + r(x)\right) - \frac{1}{2b(x)}m_2^{[x]}(x) + o(1) \\ &\leq -m_2^{[x]}(x)/2b(x) + o(1) \quad \text{as } x \rightarrow \infty, \end{aligned}$$

owing to (2.14). Then (2.16) implies (2.17) for all sufficiently large  $x$  and the proof is complete.  $\square$

### 2.2.2 Positive recurrence in the case $xm_1(x) \rightarrow \infty$ with possibly infinite second moments

Let, for some decreasing function  $r(x)$  and  $x_0 > 0$ ,

$$m_1^{[x/2]}(x) \leq -r(x) \quad \text{for all } x \geq x_0. \quad (2.42)$$

Define

$$W(x) := \int_0^x \min\left(1, \frac{1}{yr(y)}\right) dy.$$

If  $xr(x)$  is increasing then

$$W(x) \geq \frac{1}{r(x)} \quad \text{ultimately in } x; \quad (2.43)$$

Consider the test function  $L(x) = 0$  for all  $x \leq 0$  and

$$L(x) := \int_0^x W(y) dy \quad \text{for } x > 0.$$

**Theorem 2.5.** *Let the condition (2.42) hold for some decreasing  $r(x)$  such that  $xr(x)$  is increasing to infinity. Let, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\xi(x)W(\xi(x)), \xi(x) > x/2\} \rightarrow 0, \quad (2.44)$$

$$\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x/2\} = o(xr(x)). \quad (2.45)$$

*Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is positive recurrent.*

Since  $W(x) \geq 1/r(x)$ , the conditions (2.44) and (2.45) are fulfilled if, for example,

$$\text{the family } \{|\xi(x)|W(|\xi(x)|), x \geq 0\} \text{ is uniformly integrable.} \quad (2.46)$$

**Corollary 2.6.** *Let, for some  $\alpha \in (0, 1)$ ,  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} \leq -\varepsilon/x^\alpha \text{ for all } x > x_0.$$

*Let also, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\xi^{1+\alpha}(x), \xi(x) > x/2\} \rightarrow 0, \quad (2.47)$$

$$\mathbb{E}\{\xi^2(x), |\xi(x)| \leq x/2\} = o(x^{1-\alpha}), \quad (2.48)$$

*Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is positive recurrent.*

Notice that both (2.48) and (2.47) hold provided that the family of random variables  $\{|\xi(x)|^{1+\alpha}, x > 0\}$  is uniformly integrable.

*Proof of Corollary 2.6.* follows if we take  $r(x) = \varepsilon/2x^\alpha$ , then  $L(x) \sim cx^{1-\alpha}$ .  $\square$

*Proof of Theorem 2.5.* By the construction,  $L'(x) = W(x)$  and

$$L''(x) = \min\left(1, \frac{1}{xr(x)}\right) > 0 \text{ is decreasing;} \quad (2.49)$$

in particular,  $W(x)$  is a concave function.

Let us prove that the mean drift of  $L(x)$  is negative and bounded away from zero for all sufficiently large  $x$ . We start with the following upper bound

$$\begin{aligned} \mathbb{E}L(x + \xi(x)) - L(x) &\leq \mathbb{E}\{L(x + \xi(x)) - L(x); |\xi(x)| \leq x/2\} \\ &\quad + \mathbb{E}\{L(x + \xi(x)); \xi(x) > x/2\} \\ &=: E_1(x) + E_2(x). \end{aligned} \quad (2.50)$$

Let us estimate the first term on the right via Taylor's theorem:

$$\begin{aligned} E_1(x) &= L'(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} + \frac{1}{2}\mathbb{E}\{L''(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq x/2\} \\ &= W(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} + \frac{1}{2}\mathbb{E}\{W'(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq x/2\}, \end{aligned}$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ . Since  $W'$  decreases and

$$W'(x/2) = \frac{2}{xr(x/2)} \leq \frac{2}{xr(x)},$$

we deduce

$$\begin{aligned} E_1(x) &\leq W(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} + \frac{1}{2}W'(x/2)\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x/2\} \\ &\leq W(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} + \frac{1}{xr(x)}\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x/2\}. \end{aligned}$$

The condition (2.45) allows to conclude that

$$E_1(x) \leq W(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} + o(1) \quad \text{as } x \rightarrow \infty. \quad (2.51)$$

In order to estimate the second expectation on the right of (2.50) first notice that

$$\begin{aligned} L(3x) &= L(x) + W(x)2x + W'(x + \theta 2x)2x^2 \\ &\leq L(x) + W(x)2x + W'(x)2x^2, \end{aligned}$$

because  $W'(x)$  decreases. Since  $L(x) \leq xW(x)$  and  $W(x) \geq 1/r(x)$ ,

$$L(3x) \leq 3xW(x) + 2x/r(x) \leq 5xW(x).$$

Therefore,

$$\begin{aligned} E_2(x) &\leq \mathbb{E}\{L(3\xi(x)); \xi(x) > x/2\} \\ &\leq 5\mathbb{E}\{\xi(x)W(\xi(x)); \xi(x) > x/2\} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \end{aligned} \quad (2.52)$$

due to the condition (2.44). Substituting (2.51) and (2.52) into (2.50) we get, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{E}L(x + \xi(x)) - L(x) &\leq W(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq x/2\} + o(1) \\ &\leq -W(x)r(x) + o(1), \end{aligned}$$

by (2.42). The inequality  $W(x) \geq 1/r(x)$  implies that the drift of  $L(X_n)$  is negative and bounded away from zero for all sufficiently large  $x$ .  $\square$

## 2.3 Non-positivity

For any Borel set  $B \subset \mathbb{R}$  denote by  $\tau_B$  the first entering of  $X_n$  to  $B$ ,

$$\tau_B := \inf\{n \geq 1 : X_n \in B\},$$

with standard convention  $\inf \emptyset = \infty$ . In this section we are interested in conditions which provides non-positivity of a Markov chain  $X_n$ , that is, when it is not true that  $\mathbb{E}_x \tau_{(-\infty, x_*]}$  is finite for  $x > x_*$ ; more precisely, when there is  $x_*$  such that either the chain is transient,  $\mathbb{P}_x\{\tau_{(-\infty, x_*]} = \infty\} > 0$ , or null-recurrent,  $\mathbb{P}_x\{\tau_{(-\infty, x_*]} < \infty\} = 1$  and  $\mathbb{E}_x \tau_{(-\infty, x_*]} = \infty$  for all  $x > x_*$ .

As follows from the condition (2.2) for positive recurrence of diffusion processes, the condition for non-positivity of diffusion processes is opposite one, that is,

$$\text{the function } \frac{1}{m_2(x)} e^{\int_0^x \frac{2m_1(y)}{m_2(y)} dy} \quad \text{is not integrable at infinity.} \quad (2.53)$$

Fix an increasing function  $s(x) \leq x/2$ . Let

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \geq -r(x) \quad \text{for all } x > x_0, \quad (2.54)$$

for a decreasing function  $r(x) > 0$ . In the next theorem we show that the chain  $X_n$  is not positive recurrent provided that the function

$$e^{-R(x)} = e^{-\int_0^x r(y)dy} \text{ is not integrable at infinity,} \quad (2.55)$$

which is motivated by the condition (2.53) for non-positivity of diffusion processes. It turns out to be very close to guarantee non-positivity of  $X_n$  but we still need some additional technical conditions on  $r(x)$  and on the increasing function

$$W(x) := e^{R(x)} \int_0^x e^{-R(y)} dy.$$

Proving non-positivity seems to be the most difficult problem in this chapter. We know two different approaches, one is due to Lamperti [50] and another one goes back to Asymont et al. [51]. Here we follow the first approach significantly improving the non-positivity result of [50].

**Theorem 2.7.** *Let the conditions (2.54) and (2.55) hold for some decreasing and differentiable function  $r(x)$ , so  $W(x)$  is twice differentiable. Let  $W(x)$  be convex and satisfy the conditions (2.19) and (2.20). Let*

$$\mathbb{E}\{|\xi(x)|^3, \xi(x) \in [-s(x), 0]\} = o(x^2/W(x)), \quad (2.56)$$

$$\mathbb{P}\{\xi(x) \leq -s(x)\} = o(1/xW(x)) \text{ as } x \rightarrow \infty, \quad (2.57)$$

and

$$m_1(x) \geq -c_1/x, \quad c_1 > 0, \text{ for all } x > x_0, \quad (2.58)$$

$$c_2 := \sup_{x>0} m_2(x) < \infty, \quad (2.59)$$

$$\liminf_{x \rightarrow \infty} m_2^{[s(x)]}(x) > 0. \quad (2.60)$$

Then there is  $x_*$  such that  $\mathbb{E}_x \tau_{(-\infty, y]} = \infty$  for all  $x > y > x_*$ .

The conditions (2.56) and (2.57) are fulfilled if, for example,

$$\text{the family } \{\xi^-(x)W(\xi^-(x)), x \geq 0\} \text{ is uniformly integrable.} \quad (2.61)$$

**Corollary 2.8.** *Let, for some  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\}} \geq -\frac{1-\varepsilon}{x} \text{ for all } x > x_0.$$

Let the conditions (2.58)–(2.60) hold,

$$\mathbb{E}\{\xi^3(x), \xi(x) \in [-s(x), 0]\} = o(x), \quad (2.62)$$

$$\mathbb{P}\{\xi(x) \leq -s(x)\} = o(1/x^2) \text{ as } x \rightarrow \infty, \quad (2.63)$$

Then there is  $x_*$  such that  $\mathbb{E}_x \tau_{(-\infty, x_*]} = \infty$  for all  $x > x_*$ .

Notice that both (2.62) and (2.63) hold provided that the family of random variables  $\{(\xi^-(x))^2, x > 0\}$  is uniformly integrable.

*Proof of Corollary 2.8.* follows if we take  $r(x) = (1 - \varepsilon/2)/x$ , then  $R(x) = (1 - \varepsilon/2) \log x$ ,  $e^{-R(x)} = 1/x^{1-\varepsilon/2}$ ,  $W(x) = 2x/\varepsilon$ , and  $L(x) = x^2/\varepsilon$ .  $\square$

**Corollary 2.9.** *Let, for some  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq x\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}} \geq -\frac{1}{x} - \frac{1}{x \log x} - \dots - \frac{1}{x \log x \dots \log_{(m-1)} x} - \frac{1 - \varepsilon}{x \log x \dots \log_{(m)} x}$$

for all sufficiently large  $x$ . Let the conditions (2.58)–(2.60) hold,

$$\mathbb{E}\{\xi(x)^3; \xi(x) \in [-s(x), 0]\} = o(x/\log x \dots \log_{(m)} x), \quad (2.64)$$

and

$$\mathbb{P}\{\xi(x) \leq -s(x)\} = o(1/x^2 \log x \dots \log_{(m)} x). \quad (2.65)$$

Then there is  $x_*$  such that  $\mathbb{E}_x \tau_{(-\infty, x_*]} = \infty$  for all  $x > x_*$ .

This result was first established by Menshikov et al. [51] under the condition that moments of order  $x^2 \log^{2+\delta} x$  are bounded. Notice that both (2.64) and (2.65) hold provided that the family

$$\{(\xi^-(x))^2 \log \xi^-(x) \dots \log_{(m)} \xi^-(x), x > 0\} \text{ is uniformly integrable.}$$

*Proof of Corollary 2.9.* Consider

$$r(x) := \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m-1)} x} + \frac{1 - \varepsilon/2}{x \log x \dots \log_{(m)} x};$$

then

$$\begin{aligned} R(x) &= \log x + \log \log x + \dots + \log_{(m)} x + (1 - \varepsilon/2) \log_{(m+1)} x, \\ e^{-R(x)} &= \frac{1}{x \cdot \log x \cdot \dots \cdot \log_{(m-1)} x \cdot \log_{(m)}^{1-\varepsilon/2} x}, \\ W(x) &= \frac{2}{\varepsilon} x \log x \cdot \dots \cdot \log_{(m-1)} x \cdot \log_{(m)} x, \\ L(x) &\sim \frac{1}{\varepsilon} x^2 \log x \cdot \dots \cdot \log_{(m-1)} x \cdot \log_{(m)} x. \end{aligned}$$

$\square$

*Proof of Theorem 2.7.* Consider the non-negative test function  $L(x)$  defined to be zero on the negative half-line and

$$L(x) := \int_0^x W(y) dy \quad \text{for all } x \geq 0.$$

First let us prove that the mean drift of  $L(x)$  is positive and bounded away from zero for all sufficiently large  $x$ , more precisely, let us prove that for some  $x_*$  and  $\varepsilon > 0$ ,

$$\mathbb{E}\{L(x + \xi(x)) - L(x); \xi(x) \leq s(x)\} \geq \varepsilon \quad \text{for all } x > x_*. \quad (2.66)$$

Having this in mind, we analyse Taylor's theorem for the function  $L$ , with the Lagrange form of the remainder, here  $x, x + y > 0$ :

$$\begin{aligned} L(x + y) - L(x) &= L'(x)y + L''(x)\frac{y}{2} + L'''(x + \theta y)\frac{y^3}{6} \\ &= W(x)y + r(x)W(x)\frac{y^2}{2} + \frac{y^2}{2} + W''(x + \theta y)\frac{y^3}{6}, \end{aligned} \quad (2.67)$$

where  $0 \leq \theta = \theta(x, y) \leq 1$ . Since  $W(x)$  is assumed to be convex,  $W'' \geq 0$  and hence

$$L(x + y) - L(x) \geq W(x)y + r(x)W(x)\frac{y^2}{2} + \frac{y^2}{2} \quad \text{for all } y > 0. \quad (2.68)$$

Substituting (2.35) into (2.67) we get

$$\begin{aligned} L(x + y) - L(x) &\geq W(x)y + r(x)W(x)\frac{y^2}{2} + \frac{y^2}{2} - c_4 \frac{W(x)}{x^2} |y|^3 \quad \text{for all } y \in [-x/2, 0]. \end{aligned} \quad (2.69)$$

Now we are ready to estimate the mean drift of  $L(X_n)$ . Non-negativity of  $L$  yields the following lower bound

$$\begin{aligned} \mathbb{E}L(x + \xi(x)) - L(x) &\geq -L(x)\mathbb{P}\{\xi(x) \leq -s(x)\} \\ &\quad + \mathbb{E}\{L(x + \xi(x)) - L(x); \xi(x) \in [-s(x), 0]\} \\ &\quad + \mathbb{E}\{L(x + \xi(x)) - L(x); \xi(x) > 0\}. \end{aligned} \quad (2.70)$$

It follows from (2.68) that

$$\begin{aligned} &\mathbb{E}\{L(x + \xi(x)) - L(x); \xi(x) \in [0, s(x)]\} \\ &\geq W(x)\mathbb{E}\{\xi(x); \xi(x) \in [0, s(x)]\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); \xi(x) \in [0, s(x)]\} \\ &\quad + \frac{1}{2}\mathbb{E}\{\xi^2(x); \xi(x) \in [0, s(x)]\}. \end{aligned} \quad (2.71)$$

It follows from (2.69) that

$$\begin{aligned} &\mathbb{E}\{L(x + \xi(x)) - L(x); \xi(x) \in [-s(x), 0]\} \\ &\geq W(x)\mathbb{E}\{\xi(x); \xi(x) \in [0, x]\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); \xi(x) \in [-s(x), 0]\} \\ &\quad + \frac{1}{2}\mathbb{E}\{\xi^2(x); \xi(x) \in [-s(x), 0]\} - c_4 \frac{W(x)}{x^2} \mathbb{E}\{|\xi(x)|^3; \xi(x) \in [-s(x), 0]\} \\ &\geq W(x)\mathbb{E}\{\xi(x); \xi(x) \in [-s(x), 0]\} + \frac{1}{2}r(x)W(x)\mathbb{E}\{\xi^2(x); \xi(x) \in [-s(x), 0]\} \\ &\quad + \frac{1}{2}\mathbb{E}\{\xi^2(x); \xi(x) \in [-s(x), 0]\} + o(1) \quad \text{as } x \rightarrow \infty, \end{aligned} \quad (2.72)$$

due to the condition (2.56). Finally, it follows from (2.57) and inequality  $L(x) \leq xW(x)$  that the first term on the right of (2.70) goes to zero as  $x \rightarrow \infty$ . Together with the upper bounds (2.71) and (2.72) it implies that

$$\begin{aligned} & \mathbb{E}\{L(x + \xi(x)) - L(x); \xi(x) \leq s(x)\} \\ & \geq W(x)m_1^{[s(x)]}(x) + \frac{1}{2}r(x)W(x)m_2^{[s(x)]}(x) + \frac{1}{2}m_2^{[s(x)]}(x) + o(1) \\ & \geq m_2^{[s(x)]}(x)/2 + o(1) \quad \text{as } x \rightarrow \infty, \end{aligned}$$

owing to (2.54). Then (2.60) implies (2.66) for all sufficiently large  $x$ , say for  $x > x_*$ .

Let  $x_0 > x_*$  and  $x_1 > x_0 + s(x_0)$ . Consider an auxiliary Markov chain  $Y_n$  living in  $(-\infty, x_1 + s(x_1)]$  whose jumps  $\eta(x)$  satisfy

$$x + \eta(x) = \min\{x + \xi(x), x_1 + s(x_1)\},$$

so the trajectories of  $X_n$  and  $Y_n$  coincide till the first time when  $X_n$  leaves the set  $(-\infty, x_1]$ . By the construction of  $Y_n$  and because  $s(x)$  increases, we also have

$$\mathbb{E}\{L(x + \eta(x)) - L(x); \eta(x) \leq s(x)\} \geq \varepsilon \quad \text{for all } x \in (x_*, x_1]. \quad (2.73)$$

Consider the following stopping time:

$$\begin{aligned} \theta &:= \min\{n \geq 1 : Y_n \leq x_* \text{ or } Y_n > x_1\} \\ &= \min\{n \geq 1 : X_n \leq x_* \text{ or } X_n > x_1\}, \end{aligned}$$

and define one more auxiliary Markov chain  $Z_n$  which equals  $Y_n$  for all  $n \leq \theta$  and  $Z_n = Y_\theta$  for all  $n > \theta$ ; the Markov chain  $L(Z_n)$  is a submartingale. It follows from (2.73) that

$$\mathbb{E}\{\theta \mid Y_0 = x_0\} \leq \frac{L(x_1) - L(x_0)}{\varepsilon} < \infty.$$

Then, since the submartingale  $L(Z_n)$  is bounded,

$$\mathbb{E}\{L(Z_\theta) \mid Y_0 = x_0\} \geq \mathbb{E}\{L(Z_0) \mid Y_0 = x_0\} = L(x_0).$$

On the other hand,

$$\begin{aligned} & \mathbb{E}\{L(Z_\theta) \mid Y_0 = x_0\} \\ & \leq L(x_*)\mathbb{P}\{Z_\theta \leq x_* \mid Y_0 = x_0\} + L(x_1 + s(x_1))\mathbb{P}\{Z_\theta > x_1 \mid Y_0 = x_0\} \\ & \leq L(x_*) + L(x_1 + s(x_1))\mathbb{P}\{Z_\theta > x_1 \mid Y_0 = x_0\}. \end{aligned}$$

Therefore,

$$\mathbb{P}\{Z_\theta > x_1 \mid Y_0 = x_0\} \geq \frac{L(x_0) - L(x_*)}{L(x_1 + s(x_1))}.$$

The conditions (2.19) and (2.20) imply (2.32) which in its turn yields that  $L(2x) \leq 2c_3L(x)$  for all  $x > 0$ , hence

$$\mathbb{P}\{Z_\theta > x_1 \mid Y_0 = x_0\} \geq \frac{L(x_0) - L(x_*)}{2c_3L(x_1)}.$$



So, for all  $x_1 > x_0 + s(x_0)$ ,

$$\mathbb{P}\{X_\theta > x_1 \mid X_0 = x_0\} \geq \frac{L(x_0) - L(x_*)}{2c_3 L(x_1)}; \quad (2.74)$$

in words, starting at point  $x_0$ , the chain  $X_n$  exceeds the level  $x_1$  before touching the set  $(-\infty, x_*]$  with probability not less than the ratio on the right of (2.74).

Consider now a starting state  $x_1 > 2x_*$ , the stopping time

$$\tau = \tau_{(-\infty, x_1/2]} = \min\{n : X_n \leq x_1/2\},$$

and stopped Markov chain  $\hat{X}_n = X_{n \wedge \tau}$  with initial state  $\hat{X}_0 = x_1$  and with jumps  $\hat{\xi}(x)$ ; we have  $\hat{\xi}(x) = \xi(x)$  for all  $x > x_1/2$ . Denote  $\hat{m}_1(x) := \mathbb{E}\hat{\xi}(x)$ , by the condition (2.58) we have

$$\hat{m}_1(x) \geq -2c_1/x_1 \quad \text{for all } x \in \mathbb{R}. \quad (2.75)$$

Given  $\hat{X}_0 = x_1$ , the process

$$M_n := X_n - x_1 - \sum_{k=0}^{n-1} \hat{m}_1(\hat{X}_k) = \sum_{k=0}^{n-1} (\xi(\hat{X}_k) - \hat{m}_1(\hat{X}_k))$$

is a square integrable—by (2.59)—martingale,  $M_0 = 0$ . Then, by (2.75),

$$\hat{X}_n = x_1 + M_n + \sum_{k=0}^{n-1} \hat{m}_1(\hat{X}_k) \geq x_1 + M_n - 2c_1 n/x_1,$$

which implies, for  $n \leq x_1^2/8c_1$ ,

$$\begin{aligned} \mathbb{P}\{\hat{X}_n \leq x_1/2 \mid \hat{X}_0 = x_1\} &= \mathbb{P}\{M_n \leq -x_1/2 + 2c_1 n/x_1\} \\ &\leq \mathbb{P}\{M_n \leq -x_1/4\} \\ &\leq 16 \frac{\mathbb{E}M_n^2}{x_1^2} \leq 16c_2 \frac{n}{x_1^2}, \end{aligned}$$

owing to Chebyshev's inequality and upper bound for square integrable martingale  $\mathbb{E}M_n^2 \leq c_2 n$  which follows from (2.59). Hence, for  $n \leq x_1^2/32c_2$ ,

$$\mathbb{P}\{\hat{X}_n > x_1/2 \mid \hat{X}_0 = x_1\} \geq 1/2.$$

Since  $\hat{X}_n$  is  $X_n$  stopped when entering  $(-\infty, x_1/2]$ , the event  $\hat{X}_n > x_1/2$  yields  $\tau \geq n$ , so

$$\mathbb{P}\{\tau_{(-\infty, x_1/2]} \geq x_1^2/32c_2 \mid X_0 = x_1\} \geq 1/2.$$

So, starting at point  $x_0$ , with probability estimated from below in (2.74),  $X_n$  reaches level  $x_1$  before entering  $(-\infty, x_*]$ , and then does not drop below level  $x_1$  within time interval of length  $[x_1^2/32c_2]$  with probability at least  $1/2$ . Therefore,

$$\mathbb{P}\{\tau_{(-\infty, x_*)} \geq x_1^2/32c_2 \mid X_0 = x_0\} \geq \frac{L(x_0) - L(x_*)}{4c_3 L(x_1)}.$$

Thus,

$$\mathbb{P}\{\tau_{(-\infty, x_*]} \geq j \mid X_0 = x_0\} \geq \frac{L(x_0) - L(x_*)}{4c_3 L(\sqrt{32c_2 j})} \geq c_4 \frac{L(x_0) - L(x_*)}{L(\sqrt{j})}, \quad c_4 < \infty.$$

It remains to prove that the function  $1/L(\sqrt{x})$  is not integrable. Indeed, since  $L(y) \leq yW(y)$ ,

$$\int_1^\infty \frac{1}{L(\sqrt{x})} dx = 2 \int_1^\infty \frac{y}{L(y)} dy \geq 2 \int_1^\infty \frac{1}{W(y)} dy.$$

Taking into account that

$$\frac{1}{W(y)} = \frac{e^{-R(y)}}{\int_0^y e^{-R(z)} dz} = \frac{d}{dy} \log \int_0^y e^{-R(z)} dz,$$

we conclude non-integrability of  $1/L(\sqrt{x})$  from (2.55). Hence  $\mathbb{E}\tau_{(-\infty, x_*]}$  cannot be finite.  $\square$

## 2.4 Recurrence and null recurrence

### 2.4.1 Recurrence

Assume that, for some decreasing function  $r(x) \downarrow 0$ ,

$$\frac{2m_1^{[x]}(x)}{m_2^{[x]}(x)} \leq r(x) \quad \text{for all } x > x_0. \quad (2.76)$$

The basic condition in the next theorem is that the function

$$e^{-R(x)} = e^{-\int_0^x r(y) dy} \quad \text{is not integrable at infinity,} \quad (2.77)$$

it is motivated by the recurrence condition (2.3) for diffusion processes and turns out to be very close to guarantee recurrence of  $X_n$ . Similar to positive recurrence, recurrence of a Markov chain is more difficult thing to prove than that of a diffusion process and it requires some additional regularity conditions on  $r(x)$  and moment-like conditions on jumps.

In the next theorem we formulate conditions for recurrence in terms of a decreasing function  $\tilde{r}(x)$  dominating  $r(x)$ ,  $\tilde{r}(x) > r(x)$ , such that the function  $e^{-\tilde{R}(x)}$  is still not integrable where

$$\tilde{R}(x) := \int_0^x \tilde{r}(y) dy. \quad (2.78)$$

Consider the function  $\tilde{L}(x)$  which is zero for negative  $x$  and

$$\tilde{L}(x) := \int_0^x e^{-\tilde{R}(y)} dy \quad \text{for all } x \geq 0,$$

which is an unboundedly increasing function because  $e^{-\tilde{R}(x)}$  is assumed to be non-integrable. When we apply the next general theorem to particular regular functions  $r$ 's in Corollaries 2.11 and 2.12 below, we need to choose  $\tilde{r}$  sufficiently greater than  $r$  in order to increase the difference  $\tilde{r} - r$  and to satisfy conditions (2.80) and (2.81); on the other hand larger function  $\tilde{r}(x)$  produces smaller values of  $e^{-\tilde{R}(x)}$ , so the choice of a suitable  $\tilde{r}$  is a rather delicate task in each particular case.

**Theorem 2.10.** *Let the condition (2.76) hold. Let, as  $x \rightarrow \infty$ ,*

$$\tilde{r}'(x) = O(1/x^2), \quad (2.79)$$

$$\mathbb{E}\{\xi^3(x); \xi(x) \in [0, x]\} = o(x^2(\tilde{r}(x) - r(x))m_2^{[x]}(x)) \quad (2.80)$$

$$\mathbb{E}\{\tilde{L}(\xi(x)); \xi(x) \geq x\} = o\left((\tilde{r}(x) - r(x))e^{-\tilde{R}(x)}m_2^{[x]}(x)\right). \quad (2.81)$$

*Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is recurrent.*

**Corollary 2.11.** *Let, for some  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq x\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}} \leq \frac{1 - \varepsilon}{x} \quad \text{for all } x > x_0.$$

*Let, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\xi^3(x); \xi(x) \in [0, x]\} = o(x), \quad (2.82)$$

$$\mathbb{E}\{\xi^{\varepsilon/2}(x); \xi(x) \geq x\} = o(1/x^{2-\varepsilon/2}). \quad (2.83)$$

*Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is recurrent.*

Notice that both (2.82) and (2.83) hold provided that the family of random variables  $\{(\xi^+(x))^2, x > 0\}$  is uniformly integrable.

*Proof of Corollary 2.11.* follows if we take  $\tilde{r}(x) = (1 - \varepsilon/2)/x$  which dominates  $r(x)$ , then  $\tilde{R}(x) = (1 - \varepsilon/2) \log x$ ,  $e^{-\tilde{R}(x)} = 1/x^{1-\varepsilon/2}$ , and  $\tilde{L}(x) = 2x^{\varepsilon/2}/\varepsilon$ .  $\square$

**Corollary 2.12.** *Let, for some  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq x\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}} \leq \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m-1)} x} + \frac{1 - \varepsilon}{x \log x \dots \log_{(m)} x}$$

*for all sufficiently large  $x$ . Let, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\xi(x)^3; \xi(x) \in [0, x]\} = o\left(\frac{x}{\log x \dots \log_{(m)} x}\right), \quad (2.84)$$

*and*

$$\mathbb{E}\{\log_{(m)}^{\varepsilon/2} \xi(x); \xi(x) > x\} = o\left(\frac{1}{x^2 \cdot \log x \dots \log_{(m-1)} x \cdot \log_{(m)}^{1-\varepsilon/2} x}\right). \quad (2.85)$$

*Then there exists an  $x_*$  such that the set  $(-\infty, x_*]$  is recurrent.*

This result was first established by Menshikov et al. [51] under the condition that moments of order  $x^2 \log^{2+\delta} x$  are bounded. Notice that both (2.84) and (2.85) hold provided that

$$\sup_x \mathbb{E}\{(\xi(x) \log \xi(x) \dots \log_{(m)} \xi(x))^2; \xi(x) > 0\} < \infty.$$

*Proof of Corollary 2.11.* Consider

$$\tilde{r}(x) := \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m-1)} x} + \frac{1 - \varepsilon/2}{x \log x \dots \log_{(m)} x};$$

then

$$\begin{aligned} \tilde{R}(x) &= \log x + \log \log x + \dots + \log_{(m)} x + (1 - \varepsilon/2) \log_{(m+1)} x, \\ r(x) - \tilde{r}(x) &= O\left(\frac{1}{x \log x \dots \log_{(m)} x}\right), \\ e^{-\tilde{R}(x)} &= \frac{1}{x \cdot \log x \cdot \dots \cdot \log_{(m-1)} x \log_{(m)}^{1-\varepsilon/2} x}, \\ \tilde{L}(x) &= \frac{2}{\varepsilon} \log_{(m)}^{\varepsilon/2} x. \end{aligned}$$

□

*Proof of Theorem 2.10.* is based on the standard approach of construction of a non-negative increasing unbounded test function whose mean drift is non-positive outside the set  $(-\infty, x_*]$ .

Let us prove that the increasing Lyapunov function  $\tilde{L}(x)$  is appropriate. Since  $\tilde{L}(x)$  is increasing, for  $x > 0$ ,

$$\begin{aligned} &\mathbb{E}\tilde{L}(x + \xi(x)) - \tilde{L}(x) \\ &\leq \mathbb{E}\{\tilde{L}(x + \xi(x)) - \tilde{L}(x); |\xi(x)| \leq x\} + \mathbb{E}\{\tilde{L}(x + \xi(x)); \xi(x) > x\} \\ &\leq \tilde{L}'(x)m_1^{[x]}(x) + \frac{1}{2}\tilde{L}''(x)m_2^{[x]}(x) + \frac{1}{6}\tilde{L}'''(x + \theta\xi(x))\mathbb{E}\{\xi^3(x); \xi(x) \leq x\} \\ &\quad + \mathbb{E}\{\tilde{L}(2\xi(x)); \xi(x) > x\}, \end{aligned} \tag{2.86}$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ , by Taylor's theorem with the remainder in the Lagrange form.

The function  $\tilde{L}'(x) = e^{-\tilde{R}(x)}$  is decreasing, so  $\tilde{L}(x)$  is concave on  $\mathbb{R}^+$ . Thus  $\tilde{L}(2x) \leq 2\tilde{L}(x)$  and hence the fourth term on the right side of (2.86) may be bounded as follows:

$$\mathbb{E}\{\tilde{L}(2\xi(x)); \xi(x) > x\} = o\left((r(x) - \tilde{r}(x))e^{-\tilde{R}(x)}m_2^{[x]}(x)\right), \tag{2.87}$$

owing the condition (2.81).

By the construction,  $\tilde{L}'(x) = e^{-\tilde{R}(x)}$  and  $\tilde{L}''(x) = -\tilde{r}(x)e^{-\tilde{R}(x)}$ , so the sum of the first and second terms on the right side of (2.86) equals

$$\frac{1}{2}e^{-\tilde{R}(x)}m_2^{[x]}(x)\left(\frac{2m_1^{[x]}(x)}{m_2^{[x]}(x)} - \tilde{r}(x)\right) \leq -\frac{1}{2}e^{-\tilde{R}(x)}(\tilde{r}(x) - r(x))m_2^{[x]}(x), \quad (2.88)$$

Again by the construction of  $\tilde{L}$ ,

$$\tilde{L}'''(x) = (-\tilde{r}'(x) + \tilde{r}^2(x))e^{-\tilde{R}(x)},$$

hence  $\tilde{L}'''(x) \geq 0$  for all  $x$  and, for  $x$  and  $y > 0$ ,

$$\begin{aligned} \tilde{L}'''(x+y) &\leq (-\tilde{r}'(x+y) + \tilde{r}^2(x))e^{-\tilde{R}(x)} \\ &\leq (c_1/x^2 + \tilde{r}^2(x))e^{-\tilde{R}(x)} \\ &\leq c_2e^{-\tilde{R}(x)}/x^2, \end{aligned}$$

due to (2.79), which particularly implies  $\tilde{r}(x) = O(1/x)$ . Hence,

$$\begin{aligned} \mathbb{E}\{\tilde{L}'''(x + \theta\xi(x))\xi^3(x); \xi(x) \leq x\} &\leq \mathbb{E}\{\tilde{L}'''(x + \theta\xi(x))\xi^3(x); \xi(x) \in [0, x]\} \\ &\leq c_2 \frac{e^{-\tilde{R}(x)}}{x^2} \mathbb{E}\{\xi(x)^3; \xi(x) \in [0, x]\} \\ &= o(e^{-\tilde{R}(x)}(r(x) - \tilde{r}(x))m_2^{[x]}(x)), \quad (2.89) \end{aligned}$$

by the condition (2.80). Substituting (2.87)–(2.89) into (2.86) we finally get

$$\mathbb{E}\tilde{L}(x + \xi(x)) - \tilde{L}(x) \leq -\frac{1 + o(1)}{2}e^{-\tilde{R}(x)}(r(x) - \tilde{r}(x))m_2^{[x]}(x) \quad \text{as } x \rightarrow \infty.$$

We now see that the right hand side is negative for all sufficiently large  $x$ , say for  $x > x_*$ , and the proof is complete.  $\square$

### 2.4.2 Null recurrence

Null recurrent diffusions are those whose drift is negligible compared to diffusion.

Combining Corollaries 2.11 and 2.8 we get the following conditions for null recurrence.

**Corollary 2.13.** *Let, for some  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\left| \frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\}} \right| \leq \frac{1 - \varepsilon}{x} \quad \text{for all } x > x_0.$$

*Let the conditions (2.58)–(2.60) hold, and the family of random variables  $\{(\xi^2(x)), x > 0\}$  is uniformly integrable. Then there is  $x_*$  such that  $\tau_{(-\infty, x_*]}$  is finite a.s. but  $\mathbb{E}_x\tau_{(-\infty, x_*]} = \infty$  for all  $x > x_*$ .*

Combining Corollaries 2.12 and 2.9 we get another conditions for null recurrence.

**Corollary 2.14.** *Let, for some  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,*

$$\left| \frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq x\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq x\}} \right| \leq \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m-1)} x} + \frac{1 - \varepsilon}{x \log x \dots \log_{(m)} x}$$

*for all sufficiently large  $x$ . Let the conditions (2.58)–(2.60) hold, and the family*

$$\{(\xi(x) \log \xi(x) \dots \log_{(m)} \xi(x))^2, x > 0\} \text{ is uniformly integrable.}$$

*Then there is  $x_*$  such that  $\tau_{(-\infty, x_*]}$  is finite a.s. but  $\mathbb{E}_x \tau_{(-\infty, x_*]} = \infty$  for all  $x > x_*$ .*

## 2.5 Transience

### 2.5.1 Condition motivated by diffusions

Fix an increasing function  $s(x) \rightarrow \infty$  as  $x \rightarrow \infty$  such that  $s(x) = o(x)$ . Assume that, for some decreasing function  $r(x) > 0$ ,

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \geq r(x) \quad \text{for } x > x_0; \quad (2.90)$$

in general, this means that the drift to the right dominates the diffusion and then the Markov chain  $X_n$  is typically transient.

The main condition in the next theorem is that the function

$$e^{-R(x)} = e^{-\int_0^x r(y) dy} \quad \text{is integrable,} \quad (2.91)$$

it is motivated by the transience condition (2.4) for diffusion processes and turns out to be very close to guarantee transience of  $X_n$ . Similar to positive recurrence, transience of a Markov chain is more difficult thing to prove than that of a diffusion process and it requires some additional regularity conditions on  $r(x)$  and moment-like conditions on jumps.

**Theorem 2.15.** *Let the conditions (2.90) and (2.91) hold. Let  $r(x) = O(1/x)$  and a decreasing differentiable function  $\tilde{r}(x) \leq r(x)$  be such that*

$$\tilde{R}(x) := \int_0^x \tilde{r}(y) dy \rightarrow \infty \quad \text{as } x \rightarrow \infty, \quad (2.92)$$

$$e^{-\tilde{R}(x-s(x))} = O(e^{-\tilde{R}(x)}) \quad \text{as } x \rightarrow \infty, \quad (2.93)$$

*and the function  $e^{-\tilde{R}(x)}$  is integrable. Let, as  $x \rightarrow \infty$ ,*

$$\tilde{r}'(x) = O(1/x^2), \quad (2.94)$$

$$\mathbb{E}\{|\xi(x)|^3; \xi(x) \in [-s(x), 0]\} = o(x^2(r(x) - \tilde{r}(x))m_2^{[s(x)]}(x)), \quad (2.95)$$

*and*

$$\mathbb{P}\{\xi(x) \leq -s(x)\} = o\left((r(x) - \tilde{r}(x))e^{-\tilde{R}(x)}m_2^{[s(x)]}(x)\right). \quad (2.96)$$

If, in addition,

$$\mathbb{P}\left\{\limsup_{n \rightarrow \infty} X_n = \infty\right\} = 1, \quad (2.97)$$

then  $X_n \rightarrow \infty$  as  $n \rightarrow \infty$  with probability 1, so  $X_n$  is transient.

The condition (2.97) (which was first proposed in this framework by Lamperti [48]) can be equivalently restated as follows: for any  $N$  the exit time from the set  $[0, N]$  is finite with probability 1. In this way it is clear that, for a countable Markov chain, the irreducibility implies (2.97). For a Markov chain on general state space, the related topic is  $\psi$ -irreducibility, see [56, Sections 4 and 8].

If, for instance,  $r(x) = 1/x^\alpha$  for some  $\alpha \in (0, 1)$ , then  $e^{-R(x)} = e^{-x^{1-\alpha}/(1-\alpha)}$  and the condition (2.93) fails for  $s(x)$  growing faster than  $x^\alpha$ . Hence (2.93) allows to consider an arbitrary  $s(x)$  of order  $o(x)$  in the only case where drifts are of order  $O(1/x)$ , see corollaries below. In the next subsection we present conditions that are more appropriate for drifts characterised by the convergence  $xm_1(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

**Corollary 2.16.** *Let, for some  $\varepsilon > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\}} \geq \frac{1 + \varepsilon}{x}$$

for all sufficiently large  $x$ . Let the truncated second moments  $m_2^{[s(x)]}(x)$  be bounded away from zero and infinity,

$$\mathbb{E}\{|\xi(x)|^3; \xi(x) \in [-s(x), 0]\} = o(x) \quad \text{as } x \rightarrow \infty, \quad (2.98)$$

and

$$\mathbb{P}\{\xi(x) \leq -s(x)\} = o(1/x^2 \log^{1+\varepsilon} x) \quad \text{as } x \rightarrow \infty. \quad (2.99)$$

If also the condition (2.97) holds, then  $X_n \rightarrow \infty$  as  $n \rightarrow \infty$  with probability 1, so  $X_n$  is transient.

Notice that both (2.98) and (2.99) hold provided

$$\sup_{x>0} \mathbb{E}\{\xi^2(x) \log^{1+2\varepsilon} \xi(x); \xi(x) < 0\} < \infty.$$

*Proof of Corollary 2.16.* follows if we take

$$\tilde{r}(x) := \frac{1}{x} + \frac{1}{x \log^{1+\varepsilon} x};$$

then  $r(x) - \tilde{r}(x) = O(1/x)$ ,  $\tilde{R}(x) = \log x + (1+\varepsilon) \log \log x$ , and  $e^{-\tilde{R}(x)} = 1/x \log^{1+\varepsilon} x$ .  $\square$

**Corollary 2.17.** *Let, for some  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,*

$$\frac{2\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\}}{\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\}} \geq \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m-1)} x} + \frac{1 + \varepsilon}{x \log x \dots \log_{(m)} x}$$

*for all sufficiently large  $x$ . Let the second moments  $m_2(x)$  be bounded away from zero and infinity*

$$\mathbb{E}\{|\xi(x)|^3; \xi(x) \in [-s(x), 0]\} = o\left(\frac{x}{\log x \dots \log_{(m)} x}\right) \text{ as } x \rightarrow \infty \quad (2.100)$$

*and*

$$\mathbb{P}\{\xi(x) \leq -s(x)\} = o\left(\frac{1}{x^2 \cdot \log^2 x \dots \log_{(m)}^2 x \cdot \log_{(m+1)}^{1+\varepsilon} x}\right) \text{ as } x \rightarrow \infty \quad (2.101)$$

*If also the condition (2.97) holds, then  $X_n \rightarrow \infty$  as  $n \rightarrow \infty$  with probability 1, so  $X_n$  is transient.*

This result was first established by Menshikov et al. [51] under the condition that moments of order  $x^2 \log^{2+\delta} x$  are bounded. Conditions (2.100) and (2.101) hold if both families of random variables

$$\sup_{x>0} \mathbb{E}\{\xi^2(x) \log^2 |\xi(x)| \dots \log_{(m)}^2 |\xi(x)| \log_{(m+1)}^{1+2\varepsilon} |\xi(x)|; \xi(x) < 0\} < \infty.$$

*Proof of Corollary 2.17.* Consider

$$\tilde{r}(x) := \frac{1}{x} + \frac{1}{x \log x} + \dots + \frac{1}{x \log x \dots \log_{(m)} x} + \frac{1 + \varepsilon}{x \log x \dots \log_{(m+1)} x};$$

then

$$\begin{aligned} r(x) - \tilde{r}(x) &= O\left(\frac{1}{x \log x \dots \log_{(m)} x}\right), \\ \tilde{R}(x) &= \log x + \log \log x + \dots + \log_{(m+1)} x + (1 + \varepsilon) \log_{(m+2)} x, \end{aligned}$$

and

$$e^{-\tilde{R}(x)} = \frac{1}{x \cdot \log x \dots \log_{(m)} x \cdot \log_{(m+1)}^{1+\varepsilon} x}.$$

□

*Proof of Theorem 2.15.* is based on the standard approach of construction of a non-negative bounded test function  $L_*(x) \downarrow 0$  such that  $L_*(X_n)$  is a supermartingale with further application of Doob's convergence theorem for supermartingales.

Consider the decreasing function

$$\begin{aligned} \tilde{L}(x) &:= \int_x^\infty e^{-\tilde{R}(y)} dy \quad \text{for all } x \geq 0, \\ \tilde{L}(x) &:= \tilde{L}(0) \quad \text{for all } x < 0, \end{aligned}$$



which is well-defined due to assumption that  $e^{-\tilde{R}(x)}$  is integrable; this function is bounded,  $\tilde{L}(x) \leq \tilde{L}(0) < \infty$ .

Let us prove that the mean drift of  $\tilde{L}(x)$  is negative for all sufficiently large  $x$ . Since  $\tilde{L}(x)$  is decreasing, we have

$$\begin{aligned} \mathbb{E}\tilde{L}(x + \xi(x)) - \tilde{L}(x) &\leq \tilde{L}(0)\mathbb{P}\{\xi(x) < -s(x)\} + \mathbb{E}\{\tilde{L}(x + \xi(x)) - \tilde{L}(x); |\xi(x)| \leq s(x)\} \\ &= \tilde{L}(0)\mathbb{P}\{\xi(x) < -s(x)\} + \tilde{L}'(x)m_1^{[s(x)]}(x) + \frac{1}{2}\tilde{L}''(x)m_2^{[s(x)]}(x) \\ &\quad + \frac{1}{6}\mathbb{E}\{\tilde{L}'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}, \end{aligned}$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ , by Taylor's theorem with the remainder in the Lagrange form. By the construction,  $\tilde{L}'(x) = -e^{-\tilde{R}(x)} < 0$ ,  $\tilde{L}''(x) = \tilde{r}(x)e^{-\tilde{R}(x)} > 0$ , and

$$\tilde{L}'''(x + y) = (\tilde{r}'(x + y) - \tilde{r}^2(x + y))e^{-\tilde{R}(x+y)} < 0 \quad (2.102)$$

$$= O(e^{-\tilde{R}(x)}/x^2) \quad (2.103)$$

as  $x \rightarrow \infty$  uniformly in  $|y| \leq s(x) = o(x)$ , due to (2.94),  $\tilde{r}(x) \leq r(x) = O(1/x)$ , and (2.93). Hence,

$$\begin{aligned} \mathbb{E}\{\tilde{L}'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\} &\leq \mathbb{E}\{\tilde{L}'''(x + \theta\xi(x))\xi^3(x); \xi(x) \in [-s(x), 0]\} \\ &\leq c_1 \frac{e^{-\tilde{R}(x)}}{x^2} \mathbb{E}\{|\xi(x)|^3; \xi(x) \in [-s(x), 0]\} \\ &= o(e^{-\tilde{R}(x)}(r(x) - \tilde{r}(x))m_2^{[s(x)]}(x)), \end{aligned}$$

by the condition (2.95), and therefore,

$$\begin{aligned} \mathbb{E}\tilde{L}(x + \xi(x)) - \tilde{L}(x) &\leq \tilde{L}(0)\mathbb{P}\{\xi(x) \leq -s(x)\} - e^{-\tilde{R}(x)}\left(m_1^{[s(x)]}(x) - \frac{1}{2}\tilde{r}(x)m_2^{[s(x)]}(x)\right) \\ &\quad + o(e^{-\tilde{R}(x)}(r(x) - \tilde{r}(x))m_2^{[s(x)]}(x)) \\ &\leq \tilde{L}(0)\mathbb{P}\{\xi(x) \leq -s(x)\} - e^{-\tilde{R}(x)}\frac{m_2^{[s(x)]}(x)}{2}(1 + o(1))(r(x) - \tilde{r}(x)), \end{aligned}$$

by (2.90). Applying now the condition (2.96) we conclude that the right hand side is negative for all sufficiently large  $x$ , so there exists a sufficiently large  $x_*$  such that

$$\mathbb{E}\tilde{L}(x + \xi(x)) - \tilde{L}(x) \leq 0 \quad \text{for all } x \geq x_*.$$

Now take  $L_*(x) := \min(\tilde{L}(x), \tilde{L}(x_*))$ . Then

$$\mathbb{E}L_*(x + \xi(x)) - L_*(x) \leq \mathbb{E}\tilde{L}(x + \xi(x)) - \tilde{L}(x) \leq 0$$

for all  $x \geq x_*$  and

$$\mathbb{E}L_*(x + \xi(x)) - L_*(x) = \mathbb{E}\{\tilde{L}(x + \xi(x)) - \tilde{L}(x_*); x + \xi(x) \geq x_*\} \leq 0$$

for all  $x < x_*$ . Therefore,  $L_*(X_n)$  constitutes a positive bounded supermartingale and, by Doob's convergence theorem,  $L_*(X_n)$  has an a.s. limit as  $n \rightarrow \infty$ . Due to the condition (2.97), this limit equals  $L_*(\infty) = 0$  and the proof is complete.  $\square$

### 2.5.2 An alternative approach to transience

Again let us fix some increasing function  $s(x) = o(x)$ .

**Theorem 2.18.** *Let, for some  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \geq \frac{1 + \varepsilon}{x} \quad \text{for all } x > x_0, \quad (2.104)$$

and

$$\mathbb{P}\{\xi(x) < -s(x)\} = o(p(x)m_1^{[s(x)]}(x)) \quad \text{as } x \rightarrow \infty, \quad (2.105)$$

where a decreasing function  $p(x) > 0$  is integrable. If, in addition, the condition (2.97) holds, then  $X_n \rightarrow \infty$  as  $n \rightarrow \infty$  with probability 1, so that  $X_n$  is transient.

Clearly the condition (2.105) is weaker than (2.99).

**Corollary 2.19.** *Let, for some  $\alpha \in (0, 1)$ ,  $\varepsilon > 0$  and  $x_0 > 0$ ,*

$$\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} \geq \frac{\varepsilon}{x^\alpha} \quad \text{for all } x > x_0.$$

Let also, as  $x \rightarrow \infty$ ,

$$\mathbb{P}\{\xi^{1+\alpha}(x), \xi(x) \leq -s(x)\} = o(p(x)/x^\alpha), \quad (2.106)$$

$$\mathbb{E}\{\xi^2(x), |\xi(x)| \leq s(x)\} = o(x^{1-\alpha}), \quad (2.107)$$

where a decreasing function  $p(x) > 0$  is integrable. If, in addition, the condition (2.97) holds, then  $X_n \rightarrow \infty$  as  $n \rightarrow \infty$  with probability 1, so that  $X_n$  is transient.

Notice that both (2.107) and (2.106) hold provided that the family of random variables  $\{|\xi(x)|^{1+\alpha}, x > 0\}$  possesses an integrable majorant.

*Proof of Theorem 2.18.* Since  $p(x)$  is decreasing and integrable, by [16], there exists a continuous decreasing integrable regularly varying at infinity with index  $-1$  function  $V_1(x)$  such that  $p(x) \leq V_1(x)$ . Take

$$V(x) := \int_x^\infty V_2(y)dy, \quad \text{where} \quad V_2(x) := \int_x^\infty \frac{V_1(y)}{2y}dy.$$

By Theorem 1(a) from [23, Ch VIII, Sec 9] we know that  $V_2$  is regularly varying with index  $-1$  and  $V_2(x) \sim V_1(x)$  as  $x \rightarrow \infty$ . Since  $V_1$  is integrable, the nonnegative decreasing function  $V(x)$  is bounded,  $V(0) < \infty$ , and  $V(x)$  is slowly varying by the same reference.

Let us prove that the mean drift of  $V(x)$  is negative for all sufficiently large  $x$ . Since  $V(x)$  is decreasing, we have

$$\begin{aligned} & \mathbb{E}V(x + \xi(x)) - V(x) \\ & \leq V(0)\mathbb{P}\{\xi(x) < -s(x)\} + \mathbb{E}\{V(x + \xi(x)) - V(x); |\xi(x)| \leq s(x)\} \\ & = V(0)\mathbb{P}\{\xi(x) < -s(x)\} + V'(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} \\ & \quad + \frac{1}{2}\mathbb{E}\{V''(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq s(x)\}, \end{aligned}$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ , by Taylor's theorem with the remainder in the Lagrange form. By the construction,  $V'(x) = -V_2(x)$  and

$$V''(x+y) = \frac{V_1(x+y)}{2(x+y)} = (1+o(1)) \frac{V_1(x)}{2x}$$

as  $x \rightarrow \infty$  uniformly in  $|y| \leq s(x)$ . Hence,

$$\begin{aligned} \mathbb{E}V(x + \xi(x)) - V(x) \\ \leq V(0)\mathbb{P}\{\xi(x) \leq -s(x)\} - V_2(x)m_1^{[s(x)]}(x) + (1+o(1))\frac{V_1(x)}{2x}m_2^{[s(x)]}(x). \end{aligned}$$

The first term on the right hand side is of order  $o(V_1(x)m_1^{[s(x)]}(x))$  by (2.105) and the inequality  $q(x) \leq V_1(x)$ . And the third term is not greater than

$$(1+o(1))V_1(x)\frac{m_1^{[s(x)]}(x)}{1+\varepsilon}$$

because of the condition (2.104). Then

$$\mathbb{E}V(x + \xi(x)) - V(x) \leq -V_1(x)m_1^{[s(x)]}(x) + V_1(x)\frac{m_1^{[s(x)]}(x)}{1+\varepsilon} + o(V_1(x)m_1^{[s(x)]}(x)).$$

This yields that there exists a sufficiently large  $x_*$  such that, for all  $x \geq x_*$

$$\mathbb{E}V(x + \xi(x)) - V(x) \leq -\frac{\varepsilon}{1+2\varepsilon}m_1^{[s(x)]}(x)V_1(x).$$

Then the rest of the proof is the same as of the proof of Theorem 2.15.  $\square$

## 2.6 Return probability for transient Markov chains

In this section we consider a transient Markov chain  $X_n$  valued in  $\mathbb{R}$ , so that, for any fixed  $\hat{x} \in \mathbb{R}$ ,

$$\mathbb{P}_x\{\tau_B < \infty\} \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

where  $\tau_B := \min\{n \geq 1 : X_n \in B\}$ ,  $B := (-\infty, \hat{x}]$ . We are interested in the rate of convergence to zero of this probability as  $x \rightarrow \infty$ . It clearly depends on the asymptotic behavior of the drift of  $X_n$  at infinity.

### 2.6.1 Drift of order $1/x$

In this subsection we consider a transient Markov chain  $X_n$  whose jumps are such that

$$m_2^{[s(x)]}(x) \rightarrow b > 0 \quad \text{and} \quad m_1^{[s(x)]}(x) \sim \frac{\mu}{x} \quad \text{as } x \rightarrow \infty, \quad (2.108)$$

where a function  $s(x) = o(x)$  is increasing and  $\mu > b/2$  which corresponds to transience subject to some minor additional conditions, see Theorem 2.18. In addition, we assume that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = r(x) + o(p(x)) \quad \text{as } x \rightarrow \infty \quad (2.109)$$

for some decreasing positive function  $r(x) \rightarrow 0$  satisfying  $r(x)x \rightarrow 2\mu/b > 1$  as  $x \rightarrow \infty$  and some decreasing integrable function  $p(x) \geq 0$ . Since  $p(x)$  is decreasing and integrable,  $p(x)x \rightarrow 0$  as  $x \rightarrow \infty$ . We also assume that

$$r'(x) = O(1/x^2) \quad \text{and} \quad p'(x) = O(1/x^2). \quad (2.110)$$

Define an increasing function

$$R(x) := \int_0^x r(y) dy. \quad (2.111)$$

Since  $xr(x) \sim 2\mu/b > 1$ ,

$$R(x) \sim \frac{2\mu}{b} \log x \quad \text{as } x \rightarrow \infty.$$

Again due to  $2\mu/b > 1$  the function  $e^{-R(x)}$  is integrable at infinity. It allows to define the following decreasing function which plays the most important role in our analysis of return probability for transient Markov chain:

$$U(x) := \int_x^\infty e^{-R(y)} dy \rightarrow 0 \quad \text{as } x \rightarrow \infty; \quad (2.112)$$

this function solves the equation  $U'' + rU' = 0$ . According to our assumptions,

$$r(z) = \frac{2\mu}{b} \frac{1}{x} + \frac{\varepsilon(x)}{x},$$

where  $\varepsilon(x) \rightarrow 0$  as  $x \rightarrow \infty$ . In view of the representation theorem for slowly varying functions, there exists a slowly varying at infinity function  $\ell(x)$  such that  $e^{-R(x)} = x^{-\rho-1}\ell(x)$  and  $U(x) \sim x^{-\rho}\ell(x)/\rho$  where  $\rho = 2\mu/b - 1 > 0$ .

The main result in this subsection is the following theorem which provides lower and upper bounds for return probability of recurrent Markov chains with asymptotically zero drift described above.

**Theorem 2.20.** *Let  $X_n$  be a transient Markov chain. Let (2.108)–(2.110) be valid with  $\mu > b/2$  and, for some increasing  $s(x) = o(x)$ ,*

$$\mathbb{E}\{|\xi(x)|^3; |\xi(x)| \leq s(x)\} = o(x^2 p(x)) \quad \text{as } x \rightarrow \infty. \quad (2.113)$$

If

$$\mathbb{P}\{\xi(x) > s(x)\} = o(p(x)/x) \quad \text{as } x \rightarrow \infty, \quad (2.114)$$

then there exist a constant  $c_1 > 0$  and a level  $\hat{x}$  such that

$$\mathbb{P}_x\{\tau_B < \infty\} \geq c_1 U(x) \quad \text{for all } x.$$

If

$$\mathbb{E}\{U(x + \xi(x)); \xi(x) < -s(x)\} = o(p(x)/x)U(x) \quad \text{as } x \rightarrow \infty, \quad (2.115)$$

then there exist a constant  $c_2 < \infty$  and a level  $x_1$  such that

$$\mathbb{P}_x\{\tau_B < \infty\} \leq c_2 \frac{U(x)}{U(\hat{x})} \quad \text{for all } x > \hat{x} > x_1.$$

To prove this result, we first need some auxiliary results. We start with a decreasing Lyapunov functions needed. Consider the functions  $r_+(x) := r(x) + p(x)$  and  $r_-(x) := r(x) - p(x)$  and define

$$\begin{aligned} R_{\pm}(x) &:= \int_0^x r_{\pm}(y) dy, \\ U_{\pm}(x) &:= \int_x^{\infty} e^{-R_{\pm}(y)} dy. \end{aligned}$$

We have  $0 \leq r_-(x) \leq r(x) \leq r_+(x)$ ,  $0 \leq R_-(x) \leq R(x) \leq R_+(x)$  and  $U_-(x) \geq U(x) \geq U_+(x) > 0$ . Since

$$C_p := \int_0^{\infty} p(y) dy \quad \text{is finite,}$$

we have

$$R_{\pm}(x) = R(x) \pm C_p + o(1) \quad \text{as } x \rightarrow \infty. \quad (2.116)$$

Therefore,

$$U_{\pm}(x) \sim e^{\mp C_p} U(x) \rightarrow \infty \quad \text{as } x \rightarrow \infty. \quad (2.117)$$

Further, since  $xr_{\pm}(x) = xr(x) \pm xp(x) \rightarrow 2\mu/b > 1$ ,

$$\frac{U'_{\pm}(x)}{(xe^{-R_{\pm}(x)})'} = \frac{-e^{-R_{\pm}(x)}}{(1 - xr_{\pm}(x))e^{-R_{\pm}(x)}} \rightarrow \frac{b}{2\mu - b} \quad \text{as } x \rightarrow \infty.$$

Then L'Hospital's rule yields

$$U_{\pm}(x) \sim \frac{b}{2\mu - b} xe^{-R_{\pm}(x)} \sim \frac{be^{\mp C_p}}{2\mu - b} xe^{-R(x)} \quad \text{as } x \rightarrow \infty. \quad (2.118)$$

**Lemma 2.21.** *If (2.113) and (2.114) hold, then*

$$\mathbb{E}U_+(x + \xi(x)) - U_{\pm}(x) \geq \left(\frac{2\mu - b}{2} + o(1)\right) \frac{p(x)}{x} U_{\pm}(x) \quad \text{as } x \rightarrow \infty. \quad (2.119)$$

*If (2.113) and (2.115) hold, then*

$$\mathbb{E}U_-(x + \xi(x)) - U_{\pm}(x) \leq -\left(\frac{2\mu - b}{2} + o(1)\right) \frac{p(x)}{x} U_{\pm}(x) \quad \text{as } x \rightarrow \infty. \quad (2.120)$$

*Proof.* We start with the following decomposition:

$$\begin{aligned} \mathbb{E}U_{\pm}(x + \xi(x)) - U_{\pm}(x) &= \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); \xi(x) < -s(x)\} \\ &\quad + \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); \xi(x) > s(x)\}. \end{aligned} \quad (2.121)$$

Here the third term on the right is negative for all sufficiently large  $x$  because  $U_{\pm}$  eventually decreases and it may be bounded below as follows:

$$\begin{aligned} \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); \xi(x) > s(x)\} &\geq -U_{\pm}(x)\mathbb{P}\{\xi(x) > s(x)\} \\ &= o(p(x)/x)U_{\pm}(x), \end{aligned} \quad (2.122)$$

provided the condition (2.114) holds. Further, the first term on the right side of (2.121) is positive and possesses the following upper bound:

$$\begin{aligned} \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); \xi(x) < -s(x)\} &\leq \mathbb{E}\{U_{\pm}(x + \xi(x)); \xi(x) < -s(x)\} \\ &= o(p(x)/x)U_{\pm}(x) \\ &= o(p(x)/x)U_{\pm}(x), \end{aligned} \quad (2.123)$$

provided the condition (2.115) holds and due to the relation (2.117). To estimate the second term on the right side of (2.121), we make use of Taylor's theorem:

$$\begin{aligned} \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x)\} \\ &= U'_{\pm}(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} + \frac{1}{2}U''_{\pm}(x)\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\} \\ &\quad + \frac{1}{6}\mathbb{E}\{U'''_{\pm}(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}, \end{aligned} \quad (2.124)$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ . By the construction of  $U_{\pm}$ ,

$$U'_{\pm}(x) = -e^{-R_{\pm}(x)}, \quad U''_{\pm}(x) = r_{\pm}(x)e^{-R_{\pm}(x)} = (r(x) \pm p(x))e^{-R_{\pm}(x)}. \quad (2.125)$$

Then it follows that

$$\begin{aligned} U'_{\pm}(x)m_1^{[s(x)]}(x) + \frac{1}{2}U''_{\pm}(x)m_2^{[s(x)]}(x) \\ &= e^{-R_{\pm}(x)}\left(-m_1^{[s(x)]}(x) + (r(x) \pm p(x))\frac{m_2^{[s(x)]}(x)}{2}\right) \\ &= \frac{m_2^{[s(x)]}(x)}{2}e^{-R_{\pm}(x)}\left(-\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} + r(x) \pm p(x)\right) \\ &= \pm \frac{m_2^{[s(x)]}(x)}{2}e^{-R_{\pm}(x)}p(x)(1 + o(1)), \end{aligned}$$

by the condition (2.109). Hence, the equivalence (2.118) yields

$$U'_{\pm}(x)m_1^{[s(x)]}(x) + \frac{1}{2}U''_{\pm}(x)m_2^{[s(x)]}(x) \sim \pm m_2^{[s(x)]}(x)\frac{2\mu - b}{2b}\frac{p(x)}{x}U_{\pm}(x) \quad (2.126)$$

Finally, let us estimate the last term in (2.124). Notice that by the condition (2.110) on the derivatives of  $r(x)$  and  $p(x)$ ,

$$U_{\pm}'''(x) = (r'(x) \pm p'(x) + (r(x) \pm p(x))^2) e^{-R_{\pm}(x)} = O(1/x^3) U_{\pm}(x).$$

so that

$$|\mathbb{E}\{U_{\pm}'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}| \leq \frac{c_1}{x^3} \mathbb{E}\{|\xi^3(x)|; |\xi(x)| \leq s(x)\} U_{\pm}(x).$$

Then, in view of (2.113) and (2.117),

$$|\mathbb{E}\{U_{\pm}'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}| = o(p(x)/x) U_{\pm}(x). \quad (2.127)$$

Then it follows from (2.124), (2.126) and (2.127) that

$$\begin{aligned} \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x)\} \\ = \pm m_2^{[s(x)]}(x) \frac{2\mu - b}{2b} \frac{p(x)}{x} U_{\pm}(x) + o(p(x)/x) U_p(x). \end{aligned} \quad (2.128)$$

Substituting (2.122)—or (2.123)—and (2.128) into (2.121), we finally get the desired results.  $\square$

Lemma 2.21 implies the following result.

**Corollary 2.22.** *In conditions of Lemma 2.21, there exists an  $\hat{x}$  such that*

$$\begin{aligned} \mathbb{E}U_{-}(x + \xi(x)) - U_{-}(x) &\leq 0, \\ \mathbb{E}U_{+}(x + \xi(x)) - U_{+}(x) &\geq 0 \end{aligned}$$

for all  $x > \hat{x}$ .

*Proof of Theorem 2.20.* The process  $U_{-}(X_n)$  is bounded above by  $U_{-}(0)$ . Let  $\hat{x}$  be any level guaranteed by the last corollary,  $B = (-\infty, \hat{x}]$  and  $\tau_B := \min\{n \geq 1 : X_n \in B\}$ .

By Corollary 2.22,  $U_{-}(X_{n \wedge \tau_B})$  is a supermartingale, so hence, by the optional stopping theorem,

$$\mathbb{E}_x U_{-}(X_{\tau_B}) \leq \mathbb{E}_x U_{-}(X_0) = U_{-}(x).$$

On the other side, since  $U_{-}$  is decreasing,

$$\mathbb{E}_x U_{-}(X_{\tau_B}) \geq U_{-}(\hat{x}) \mathbb{P}_x\{\tau_B < \infty\}.$$

Therefore,

$$\mathbb{P}_x\{\tau_B < \infty\} \leq \frac{U_{-}(x)}{U_{-}(\hat{x})} \leq c_2 \frac{U(x)}{U(\hat{x})}. \quad (2.129)$$

On the other hand, the process  $U_{+}(X_{n \wedge \tau_B})$  is a bounded submartingale, then again by the optional stopping theorem,

$$\mathbb{E}_x U_{+}(X_{\tau_B}) \geq \mathbb{E}_x U_{+}(X_0) = U_{+}(x).$$

On the other side, since  $U_+$  is bounded by  $U_+(0)$ ,

$$\mathbb{E}_x U_+(X_{\tau_B}) \leq U_+(0) \mathbb{P}_x\{\tau_B < \infty\}.$$

This allows to deduce the lower bound

$$\mathbb{P}_x\{\tau_B < \infty\} \geq \frac{U_+(x)}{U_+(0)} \geq c_1 U(x). \quad (2.130)$$

The proof is complete.  $\square$

### 2.6.2 The case where $xm_1(x) \rightarrow \infty$ but $m_1(x) = o(1/\sqrt{x})$

In this subsection we consider a transient Markov chain  $X_n$  whose jumps are such that

$$m_2^{[s(x)]}(x) \rightarrow b > 0 \quad \text{and} \quad xm_1^{[s(x)]}(x) \rightarrow \infty \quad \text{as } x \rightarrow \infty, \quad (2.131)$$

for some increasing function  $s(x) = o(x)$  which implies transience subject to some minor additional conditions, see Theorem 2.18. In addition, we assume that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = r(x) + o(p(x)) \quad \text{as } x \rightarrow \infty \quad (2.132)$$

for some decreasing positive differentiable function  $r(x) \rightarrow 0$  satisfying  $r(x)x \rightarrow \infty$  as  $x \rightarrow \infty$  and some decreasing differentiable function  $p(x) \geq 0$  which is assumed to be integrable,

$$C_p := \int_0^\infty p(x) dx < \infty. \quad (2.133)$$

Since  $p(x)$  is decreasing and integrable,  $p(x)x \rightarrow 0$  as  $x \rightarrow \infty$ .

In this subsection we consider the case  $r(x) = o(1/\sqrt{x})$  as  $x \rightarrow \infty$ , more precisely,

$$r^2(x) = o(p(x)) \quad \text{as } x \rightarrow \infty. \quad (2.134)$$

We also assume that

$$|p'(x)| \leq |r'(x)|, \quad |r'(x)| = O(r^2(x)) \quad \text{as } x \rightarrow \infty. \quad (2.135)$$

In view of (2.131), the condition (2.132) is equivalent to

$$-m_1^{[s(x)]}(x) + \frac{m_2^{[s(x)]}(x)}{2} r(x) = o(p(x)) \quad \text{as } x \rightarrow \infty. \quad (2.136)$$

Define the increasing function  $R(x)$  as in (2.111). Since  $xr(x) \rightarrow \infty$ , the function  $e^{-R(x)}$  is integrable at infinity. It allows to define the decreasing function  $U(x)$  as in (2.112) which plays a key role in the next result.



**Theorem 2.23.** *Let  $X_n$  be a transient Markov chain. Let (2.131)–(2.135) be valid for some  $s(x) = o(1/r(x))$  and*

$$\sup_x \mathbb{E}\{|\xi(x)|^3; |\xi(x)| \leq s(x)\} < \infty. \quad (2.137)$$

If

$$\mathbb{P}\{\xi(x) > s(x)\} = o(p(x)r(x)) \quad \text{as } x \rightarrow \infty, \quad (2.138)$$

then there exists a constant  $c_1 > 0$  and a level  $\hat{x}$  such that

$$\mathbb{P}_x\{\tau_B < \infty\} \geq c_1 U(x) \quad \text{for all } x.$$

If

$$\mathbb{E}\{U(x + \xi(x)); \xi(x) < -s(x)\} = o(p(x)r(x))U(x) \quad \text{as } x \rightarrow \infty, \quad (2.139)$$

then there exists a constant  $c_2 < \infty$  and a level  $x_1$  such that

$$\mathbb{P}_x\{\tau_B < \infty\} \leq c_2 \frac{U(x)}{U(\hat{x})} \quad \text{for all } x > \hat{x} > x_1.$$

To prove the last theorem, we consider the same functions  $r_\pm(x)$ ,  $R_\pm(x)$  and  $U_\pm(x)$  as in the previous subsection. The only difference is that now, since  $xr_\pm(x) = xr(x) \pm xp(x) \sim xr(x)$ ,

$$\frac{U'_\pm(x)}{(\frac{1}{r_\pm(x)}e^{-R_\pm(x)})'} = \frac{-e^{-R_\pm(x)}}{(-1 - r'_\pm(x)/r_\pm^2(x))e^{-R_\pm(x)}} \rightarrow 1 \quad \text{as } x \rightarrow \infty.$$

Then L'Hospital's rule yields

$$U_\pm(x) \sim \frac{1}{r_\pm(x)}e^{-R_\pm(x)} \sim \frac{e^{\mp C_p}}{r(x)}e^{-R(x)} \quad \text{as } x \rightarrow \infty. \quad (2.140)$$

Then similarly to Lemma 2.21 the following result follows.

**Lemma 2.24.** *If the conditions (2.138) and (2.137) hold, then*

$$\mathbb{E}U_+(x + \xi(x)) - U_+(x) \geq \frac{b + o(1)}{2}p(x)r(x)U_+(x) \quad \text{as } x \rightarrow \infty. \quad (2.141)$$

If the conditions (2.139) and (2.137) hold, then

$$\mathbb{E}U_-(x + \xi(x)) - U_-(x) \leq -\frac{b + o(1)}{2}p(x)r(x)U_-(x) \quad \text{as } x \rightarrow \infty. \quad (2.142)$$

*Proof.* We start with the following decomposition:

$$\begin{aligned} \mathbb{E}U_\pm(x + \xi(x)) - U_\pm(x) &= \mathbb{E}\{U_\pm(x + \xi(x)) - U_\pm(x); \xi(x) < -s(x)\} \\ &\quad + \mathbb{E}\{U_\pm(x + \xi(x)) - U_\pm(x); |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{U_\pm(x + \xi(x)) - U_\pm(x); \xi(x) > s(x)\} \end{aligned} \quad (2.143)$$

Here the third term on the right is negative for all sufficiently large  $x$  because  $U_{\pm}$  eventually decreases and it may be bounded below as follows:

$$\begin{aligned}\mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); \xi(x) > s(x)\} &\geq -U_{\pm}(x)\mathbb{P}\{\xi(x) > s(x)\} \\ &= o(p(x)r(x))U_{\pm}(x),\end{aligned}\quad (2.144)$$

provided the condition (2.138) holds. Further, the first term on the right side of (2.143) is positive and possesses the following upper bound:

$$\begin{aligned}\mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); \xi(x) < -s(x)\} &\leq \mathbb{E}\{U_{\pm}(x + \xi(x)); \xi(x) < -s(x)\} \\ &= o(p(x)r(x))U_{\pm}(x),\end{aligned}\quad (2.145)$$

provided the condition (2.139) holds and due to the relation (2.117). To estimate the second term on the right side of (2.143), we make use of Taylor's theorem:

$$\begin{aligned}\mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x)\} \\ = U'_{\pm}(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} + \frac{1}{2}U''_{\pm}(x)\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\} \\ + \frac{1}{6}\mathbb{E}\{U'''_{\pm}(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\},\end{aligned}\quad (2.146)$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ . By the construction of  $U_{\pm}$ ,

$$U'_{\pm}(x) = -e^{-R_{\pm}(x)}, \quad U''_{\pm}(x) = r_{\pm}(x)e^{-R_{\pm}(x)} = (r(x) \pm p(x))e^{-R_{\pm}(x)}. \quad (2.147)$$

Then it follows that

$$\begin{aligned}U'_{\pm}(x)m_1^{[s(x)]}(x) + \frac{1}{2}U''_{\pm}(x)m_2^{[s(x)]}(x) \\ = e^{-R_{\pm}(x)}\left(-m_1^{[s(x)]}(x) + (r(x) \pm p(x))\frac{m_2^{[s(x)]}(x)}{2}\right) \\ = \frac{m_2^{[s(x)]}(x)}{2}e^{-R_{\pm}(x)}\left(-\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} + r(x) \pm p(x)\right) \\ = \pm \frac{m_2^{[s(x)]}(x)}{2}e^{-R_{\pm}(x)}p(x)(1 + o(1)),\end{aligned}$$

by the condition (2.132). Hence, the equivalence (2.118) yields

$$U'_{\pm}(x)m_1^{[s(x)]}(x) + \frac{1}{2}U''_{\pm}(x)m_2^{[s(x)]}(x) \sim \pm m_2^{[s(x)]}(x)p(x)r(x)U_{\pm}(x). \quad (2.148)$$

Finally, let us estimate the last term in (2.146). Notice that by the condition (2.135) on the derivatives of  $r(x)$  and  $p(x)$ ,

$$U'''_{\pm}(x) = (r'(x) \pm p'(x) + (r(x) \pm p(x))^2)e^{-R_{\pm}(x)} = O(r^2(x))e^{-R(x)}.$$

As shown in (2.13),  $R(x + s(x)) = R(x) + o(1)$  for any  $s(x) = o(1/r(x))$ . Therefore,

$$\begin{aligned}|\mathbb{E}\{U'''_{\pm}(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}| &\leq c_1r^2(x)\mathbb{E}\{|\xi^3(x)|; |\xi(x)| \leq s(x)\}e^{-R(x)} \\ &\leq c_2r^2(x)e^{-R(x)},\end{aligned}$$

owing to the boundedness (2.137) of third absolute moments. Then, in view of (2.134) and (2.140),

$$|\mathbb{E}\{U_{\pm}'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}| = o(p(x)r(x))U_{\pm}(x). \quad (2.149)$$

Then it follows from (2.146), (2.148) and (2.149) that

$$\begin{aligned} \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x)\} \\ = \pm \frac{m_2^{[s(x)]}(x)}{2} p(x)r(x)U_{\pm}(x) + o(p(x)/x)U_p(x). \end{aligned} \quad (2.150)$$

Substituting (2.144)—or (2.145)—and (2.150) into (2.143), we finally get the desired results.  $\square$

Lemma 2.24 implies the following result.

**Corollary 2.25.** *There exists an  $\hat{x}$  such that, for all  $x > \hat{x}$ ,*

$$\begin{aligned} \mathbb{E}U_{-}(x + \xi(x)) - U_{-}(x) &\leq 0, \\ \mathbb{E}U_{+}(x + \xi(x)) - U_{+}(x) &\geq 0. \end{aligned}$$

The last corollary allows to conclude the proof of Theorem 2.23 in the same way as the proof of Theorem 2.20.

### 2.6.3 General case where $xm_1(x) \rightarrow \infty$

If  $r(x)$  decreases slower than  $1/\sqrt{x}$ , then the function  $r^2(x)$  is not integrable and, since  $U_{\pm}'''(x)$  is of order  $r^3(x)U_{\pm}(x)$ , it does not possess a bound like  $p(x)r(x)U_{\pm}(x)$ . So, the last term in Taylor's expansion (2.146) is not negligible and it instead makes a significant contribution to the drift of  $U_{\pm}$ . If  $r(x)$  is sandwiched between  $1/\sqrt{x}$  and  $1/\sqrt[3]{x}$ , then we need to consider Taylor's expansion that includes the forth derivative of  $U_{\pm}$  and, consequently, the forth moments of jumps. As  $r(x)$  becomes slower decreasing, higher moments of jumps are required.

So, in this subsection we consider the same setting as in the last subsection but now we consider a general case and do not assume that  $r(x) = o(1/\sqrt{x})$ . Instead, we assume that, for some  $\gamma \in \{2, 3, 4, \dots\}$ ,

$$r^{\gamma}(x) = o(p(x)) \quad \text{as } x \rightarrow \infty \quad (2.151)$$

and

$$-m_1^{[s(x)]}(x) + \sum_{j=2}^{\gamma} (-1)^j \frac{m_j^{[s(x)]}(x)}{j!} r^{j-1}(x) = o(p(x)) \quad \text{as } x \rightarrow \infty. \quad (2.152)$$

We further assume that the function  $r(x)$  is  $\gamma$  times differentiable and satisfies the condition, as  $x \rightarrow \infty$ ,

$$r^{(k)}(x) = o(p(x)), \quad p^{(k)}(x) = o(p(x)) \quad \text{for all } 2 \leq k \leq \gamma. \quad (2.153)$$

and

$$|p'(x)| \leq |r'(x)| = o(p(x)) \quad \text{as } x \rightarrow \infty. \quad (2.154)$$

**Theorem 2.26.** *Let  $X_n$  be a transient Markov chain. Let (2.131), (2.151)–(2.154) be valid for some  $s(x) = o(1/r(x))$  and*

$$\sup_x \mathbb{E}\{|\xi(x)|^{\gamma+1}; |\xi(x)| \leq s(x)\} < \infty. \quad (2.155)$$

If

$$\mathbb{P}\{\xi(x) > s(x)\} = o(p(x)r(x)) \quad \text{as } x \rightarrow \infty, \quad (2.156)$$

then there exists a constant  $c_1 > 0$  and a level  $\hat{x}$  such that

$$\mathbb{P}_x\{\tau_B < \infty\} \geq c_1 U(x) \quad \text{for all } x.$$

If

$$\mathbb{E}\{U(x + \xi(x)); \xi(x) < -s(x)\} = o(p(x)r(x))U(x) \quad \text{as } x \rightarrow \infty, \quad (2.157)$$

then there exists a constant  $c_2 < \infty$  and a level  $x_1$  such that

$$\mathbb{P}_x\{\tau_B < \infty\} \leq c_2 U(x) \quad \text{for all } x > \hat{x} > x_1.$$

We consider the same functions  $r_{\pm}(x)$ ,  $R_{\pm}(x)$  and  $U_{\pm}(x)$  as in the previous subsection and similarly to Lemma 2.24 we get the following result.

**Lemma 2.27.** *If the conditions (2.156) and (2.155) hold, then*

$$\mathbb{E}U_+(x + \xi(x)) - U_+(x) \geq \frac{b + o(1)}{2} p(x)r(x)U_+(x) \quad \text{as } x \rightarrow \infty. \quad (2.158)$$

If the conditions (2.157) and (2.155) hold, then

$$\mathbb{E}U_-(x + \xi(x)) - U_-(x) \leq -\frac{b + o(1)}{2} p(x)r(x)U_-(x) \quad \text{as } x \rightarrow \infty. \quad (2.159)$$

*Proof.* We start with the decomposition (2.143), where the first and third terms on the right hand side possess the same bounds as in the proof of Lemma 2.24.

To estimate the second term on the right side of (2.143), we make use of Taylor's theorem:

$$\begin{aligned} & \mathbb{E}\{U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x)\} \\ &= \sum_{k=1}^{\gamma} \frac{U_{\pm}^{(k)}(x)}{k!} m_k^{[s(x)]}(x) + \mathbb{E}\left\{ \frac{U_{\pm}^{(\gamma+1)}(x + \theta\xi(x))}{(\gamma+1)!} \xi^{\gamma+1}(x); |\xi(x)| \leq s(x) \right\}, \end{aligned} \quad (2.160)$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ . By the construction of  $U_{\pm}$ ,

$$U'_{\pm}(x) = -e^{-R_{\pm}(x)}, \quad U''_{\pm}(x) = r_{\pm}(x)e^{-R_{\pm}(x)} = (r(x) \pm p(x))e^{-R_{\pm}(x)}, \quad (2.161)$$

and, for  $k = 3, \dots, \gamma + 1$ ,

$$U_{\pm}^{(k)}(x) = -(e^{-R_{\pm}(x)})^{(k-1)} = (-1)^k (r_{\pm}^{k-1}(x) + o(p(x)))e^{-R_{\pm}(x)} \quad \text{as } x \rightarrow \infty,$$

where the remainder terms in the parentheses on the right are of order  $o(p(x))$  by the condition (2.153). By the definition of  $r_{\pm}(x)$ ,

$$r_{\pm}^{k-1}(x) = (r(x) \pm p(x))^{k-1} = r^{k-1}(x) + o(p(x)) \quad \text{for all } k \geq 3,$$

which implies the relation

$$U_{\pm}^{(k)}(x) = (-1)^k (r^{k-1}(x) + o(p(x))) e^{-R_{\pm}(x)} \quad \text{as } x \rightarrow \infty. \quad (2.162)$$

It follows from the equalities (2.161) and (2.162) that

$$\begin{aligned} & \sum_{k=1}^{\gamma} \frac{U_{\pm}^{(k)}(x)}{k!} m_k^{[s(x)]}(x) \\ &= e^{-R_{\pm}(x)} \left( \sum_{k=1}^{\gamma} (-1)^k \frac{r^{k-1}(x)}{k!} m_k^{[s(x)]}(x) + o(p(x)) - p(x) \frac{m_2^{[s(x)]}(x)}{2} \right) \\ &= e^{-R_p(x)} \left( o(p(x)) - p(x) \frac{m_2^{[s(x)]}(x)}{2} \right), \end{aligned}$$

by the condition (2.152). Hence, the equivalence (2.140) yields

$$\sum_{k=1}^{\gamma} \frac{U_{\pm}^{(k)}(x)}{k!} m_k^{[s(x)]}(x) = \pm r(x) p(x) \frac{m_2^{[s(x)]}(x)}{2} U_{\pm}(x) + o(r(x) p(x)) U_{\pm}(x). \quad (2.163)$$

Owing the condition (2.153) on the derivatives of  $r(x)$  and the condition (2.151),

$$\begin{aligned} U_{\pm}^{(\gamma+1)}(x) &= (-1)^{\gamma+1} (r^{\gamma}(x) + o(p(x))) e^{-R_{\pm}(x)} \\ &= o(p(x)) e^{-R_{\pm}(x)} = o(p(x) r(x)) U_{\pm}(x). \end{aligned}$$

Then, similarly to (2.149), the last term in (2.160) possesses the following bound:

$$\begin{aligned} & \left| \mathbb{E} \left\{ \frac{U_{\pm}^{(\gamma+1)}(x + \theta \xi(x))}{(\gamma+1)!} \xi^{\gamma+1}(x); |\xi(x)| \leq s(x) \right\} \right| \\ & \leq o(p(x) r(x)) U_{\pm}(x) \mathbb{E} \{ |\xi(x)|^{\gamma+1}; |\xi(x)| \leq s(x) \} \\ & = o(p(x) r(x)) U_{\pm}(x), \end{aligned}$$

by the condition (2.155). Therefore, it follows from (2.160) and (2.163) that

$$\begin{aligned} & \mathbb{E} \{ U_{\pm}(x + \xi(x)) - U_{\pm}(x); |\xi(x)| \leq s(x) \} \\ & = \pm r(x) p(x) \frac{m_2^{[s(x)]}(x)}{2} U_{\pm}(x) + o(p(x) r(x)) U_{\pm}(x). \end{aligned}$$

Together with (2.144), (2.145), and (2.143) this completes the proof.  $\square$

Lemma 2.27 implies analogue of Corollary 2.25 which allows to conclude the proof of Theorem 2.26 in the same way as of Theorem 2.20.



## Chapter 3

# Limit theorems for transient Markov chains

Let  $r(x) > 0$  be a decreasing differentiable function such that, for some  $c > 0$ ,

$$0 \geq r'(x) \geq -cr^2(x) \quad \text{for all sufficiently large } x, \quad (3.1)$$

which yields

$$r(x) \geq \frac{1}{c_1 + cx} \quad \text{for all sufficiently large } x.$$

Then, in particular,

$$R(x) := \int_0^x r(y)dy \rightarrow \infty \quad \text{as } x \rightarrow \infty. \quad (3.2)$$

The function  $R(x)$  is concave because  $r(x)$  is decreasing. As shown in (2.11) and (2.12),

$$R(x) + \frac{h}{1+ch} \leq R(x + h/r(x)) \leq R(x) + h, \quad (3.3)$$

$$R(x) - \frac{h}{1-ch} \leq R(x - h/r(x)) \leq R(x) - h. \quad (3.4)$$

Then, as already discussed,  $1/r(x)$  is a natural step size responsible for constant increase of the function  $R(x)$  and

$$R(x + o(1/r(x))) = R(x) + o(1) \quad \text{as } x \rightarrow \infty. \quad (3.5)$$

Fix an increasing function  $s(x) \rightarrow \infty$  as  $x \rightarrow \infty$  such that  $s(x) = o(x)$ . Assume that there exist  $\hat{x}$  and  $\varepsilon > 0$  such that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \geq r(x) \geq \frac{1+\varepsilon}{x} \quad \text{for all } x > \hat{x}; \quad (3.6)$$

then the drift to the right dominates the diffusion and the corresponding Markov chain  $X_n$  is typically transient, see Theorem 2.18.

### 3.1 Quantitative analysis of escaping to infinity for transient chain

Now we produce an upper bound for the return probability for transient Markov chain which is a rough version of more precise bounds derived in Section 2.6. The main aim is to have an upper bound under conditions milder than in that section.

**Lemma 3.1.** *Assume the condition (3.6) holds for some  $r(x)$  satisfying (3.1) and  $s(x) = o(1/r(x))$ . Let, for some  $\delta \in (0, \varepsilon)$ ,*

$$\mathbb{E}\{e^{-\delta R(x+\xi(x))}; \xi(x) < -s(x)\} = o(r^2(x)e^{-\delta R(x)}m_2^{[s(x)]}(x)) \quad (3.7)$$

as  $x \rightarrow \infty$ . Then there exists  $x_*$  such that, for all  $y > x \geq 0$ ,

$$\mathbb{P}\{X_n \leq x \text{ for some } n \geq 1 \mid X_0 = y\} \leq e^{\delta(R_*(x) - R_*(y))},$$

where  $R_*(x) := \max(R(x_*), R(x))$ . In particular, for any fixed  $h > 0$ ,

$$\mathbb{P}_x\left\{X_n \leq x - \frac{h}{r(x)} \text{ for some } n \geq 1\right\} \leq e^{-\delta h/2} \quad \text{ultimately in } x.$$

*Proof.* Consider the decreasing test function  $W(x) := e^{-\delta R(x)}$  which is bounded above by 1. Let us prove that the mean drift of  $W(x)$  is negative for all sufficiently large  $x$ . Indeed, since the function  $W(x)$  decreases,

$$\begin{aligned} \mathbb{E}W(x + \xi(x)) - W(x) &\leq \mathbb{E}\{W(x + \xi(x)) - W(x); \xi(x) \leq s(x)\} \\ &\leq \mathbb{E}\{W(x + \xi(x)); \xi(x) < -s(x)\} \\ &\quad + W'(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} \\ &\quad + \frac{1}{2}\mathbb{E}\{W''(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq s(x)\} \\ &=: E_1 + E_2 + E_3, \end{aligned} \quad (3.8)$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ , by Taylor's theorem. By the condition (3.7), the first term on the right hand side is of order

$$E_1 = o(r^2(x)W(x)m_2^{[s(x)]}(x)) \quad \text{as } x \rightarrow \infty. \quad (3.9)$$

The second term on the right hand side of (3.8) equals

$$\begin{aligned} E_2 &= -\delta r(x)W(x)m_1^{[s(x)]}(x) \\ &\leq -\frac{\delta}{2}r^2(x)W(x)m_2^{[s(x)]}(x) \quad \text{ultimately in } x, \end{aligned} \quad (3.10)$$

due to (3.6).

In order to bound the third term on the right hand side of (3.8), we first notice that due to (3.1),

$$\begin{aligned} W''(x) &= \delta(\delta r^2(x) - r'(x))W(x) \\ &\leq \delta(\delta + 1 - \varepsilon)r^2(x)W(x), \end{aligned} \quad (3.11)$$



for all sufficiently large  $x$ . By (3.5),  $W(x - s(x)) \sim W(x)$  which together with decrease of  $r(x)$  and  $W(x)$  implies

$$W''(x + y) \leq \delta(1 + \delta - \varepsilon)(1 + o(1))r^2(x)W(x)$$

as  $x \rightarrow \infty$  uniformly for all  $y \geq -s(x)$ . Then

$$E_3 \leq \frac{\delta}{2} \left(1 - \frac{\varepsilon - \delta}{2}\right) r^2(x) W(x) m_2^{[s(x)]}(x) \quad (3.12)$$

for all sufficiently large  $x$ . Substituting (3.9), (3.10) and (3.12) into (3.8) we deduce, as  $x \rightarrow \infty$ ,

$$\mathbb{E}W(x + \xi(x)) - W(x) \leq -\frac{\delta(\varepsilon - \delta)}{2} r^2(x) W(x) m_2^{[s(x)]}(x).$$

Then there exists a sufficiently large  $x_*$  such that, for all  $x \geq x_*$ ,

$$\mathbb{E}W(x + \xi(x)) - W(x) < 0.$$

Now take  $W_*(x) := \min(W(x), W(x_*))$  so that  $W_*(X_n)$  constitutes positive bounded supermartingale. Hence we may apply Doob's inequality for nonnegative supermartingale and deduce that, for every  $y \geq x \geq 0$  (so that  $W_*(y) \leq W_*(x)$ ),

$$\mathbb{P}\left\{\sup_{n \geq 1} W_*(X_n) \geq W_*(x) \mid W_*(X_0) = W_*(y)\right\} \leq \frac{\mathbb{E}W_*(X_0)}{W_*(x)} = e^{\delta(R_*(x) - R_*(y))},$$

which is equivalent to the first conclusion of the lemma. Then the second conclusion follows from (3.4).  $\square$

Define

$$L(x, n) := \sum_{k=0}^{n-1} \mathbb{I}\{X_k > x\}. \quad (3.13)$$

The next lemma is devoted to the properties of  $L(x, T(t))$ , where  $T(t)$  is the first up-crossing time

$$T(t) := \min\{n \geq 1 : X_n > t\}.$$

Let  $v(z) \downarrow 0$  be a decreasing function. Denote

$$V(u) := \int_0^u \frac{1}{v(z)} dz.$$

Since the function  $1/v(z)$  increases,  $V$  is convex.

**Lemma 3.2.** *Let, for some increasing function  $s(x) > 0$  and for some  $\hat{x} \geq 0$ ,*

$$\mathbb{E}\{\xi(x); \xi(x) \leq s(x)\} \geq v(x) \quad \text{for } x > \hat{x}. \quad (3.14)$$

Then, for all  $t > y$  and  $x \in (\hat{x}, t)$ ,

$$\mathbb{E}_y L(x, T(t)) \leq V(t + s(t)) - V(x \vee y) = \int_{x \vee y}^{t+s(t)} \frac{1}{v(z)} dz. \quad (3.15)$$

Further, the family of random variables

$$\frac{1}{V(t + s(t)) - V(x)} L(x, T(t)), \quad X_0 = y, \quad y < t, \quad \hat{x} < x < t, \quad (3.16)$$

is uniformly integrable.

*Proof.* Let us consider the process

$$\tilde{X}_n := \min(X_n, t + s(t)).$$

By the construction of the process  $\tilde{X}_n$ , its stochastic behavior is identical to that of  $X_n$  to the left of  $t$ . Therefore, the stopping time

$$\tilde{T}(t) := \min\{n \geq 1 : \tilde{X}_n > t\}$$

is equal to  $T(t)$  given  $X_0 = \tilde{X}_0 = y < t$ .

Consider the following continuous test function

$$V_x(u) := V(x \vee u) = \begin{cases} V(x) & \text{if } u \leq x, \\ V(u) & \text{if } u > x. \end{cases}$$

This function is convex as  $V$  is, so Jensen's inequality yields

$$\mathbb{E}_u V_x(\tilde{X}_1) - V_x(u) \geq V'_x(u) \mathbb{E}_u(\tilde{X}_1 - u),$$

where

$$V'_x(u) = \begin{cases} 0 & \text{if } u \leq x, \\ 1/v(u) & \text{if } u > x. \end{cases}$$

Then  $\mathbb{E}_u(\tilde{X}_1 - u) \geq \mathbb{E}\{\xi(u); \xi(u) \leq s(u)\}$  for  $u \leq t$ , so hence

$$\mathbb{E}_u V_x(\tilde{X}_1) - V_x(u) \geq \begin{cases} 1 & \text{if } u \in (x, t], \\ 0 & \text{if } u \leq x, \end{cases} \quad (3.17)$$

by the condition (3.14). So,

$$Y_n := V_x(\tilde{X}_{n \wedge T(t)})$$

constitutes a submartingale with respect to the filtration  $\mathcal{F}_n := \sigma(X_k, k \leq n)$ . Hence,

$$\mathbb{E}_y Y_{T(t)} \geq V(x \vee y) + \mathbb{E}_y \sum_{k=0}^{T(t)-1} \mathbb{I}\{X_k > x\}, \quad (3.18)$$

due to the adapted version of the proof of Dynkin's formula (see, e.g. [56, Theorem 11.3.1]):

$$\begin{aligned}
\mathbb{E}_y Y_{T(t)} &= \mathbb{E}_y Y_0 + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{I}\{n \leq T(t)\} (Y_n - Y_{n-1}) \\
&= V_x(y) + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{E}\{\mathbb{I}\{n \leq T(t)\} (Y_n - Y_{n-1}) \mid \mathcal{F}_{n-1}\} \\
&= V_x(y) + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{I}\{T(t) \geq n\} \mathbb{E}\{Y_n - Y_{n-1} \mid \mathcal{F}_{n-1}\},
\end{aligned}$$

because  $\mathbb{I}\{n \leq T(t)\} \in \mathcal{F}_{n-1}$ . Hence, it follows from (3.17) that

$$\begin{aligned}
\mathbb{E}_y Y_{T(t)} &\geq V_x(y) + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{I}\{T(t) \geq n, X_{n-1} > x\} \\
&= V(x \vee y) + \mathbb{E}_y \sum_{n=1}^{T(t)} \mathbb{I}\{X_{n-1} > x\},
\end{aligned}$$

and the inequality (3.18) follows.

On the other hand,  $\tilde{X}_{T(t)} \leq t + s(t)$ , by the construction of  $\tilde{X}$ . Hence,

$$\mathbb{E}_y Y_{T(t)} \leq V_x(t + s(t)) \leq V(t + s(t)), \quad (3.19)$$

because  $x < t$ , which together with (3.18) yields

$$\mathbb{E}_y L(x, T(t)) \leq V(t + s(t)) - V(x \vee y),$$

and the proof of the upper bound (3.15) follows.

Now proceed to the proof of uniform integrability in (3.16). This assertion is equivalent to the following: as  $A \rightarrow \infty$ ,

$$\sup_{\hat{x} < x < t, y < t} \mathbb{E}_y \left\{ \frac{L(x, T(t))}{V(t + s(t)) - V(x)} - A; \frac{L(x, T(t))}{V(t + s(t)) - V(x)} > A \right\} \rightarrow 0. \quad (3.20)$$

For  $N \in \mathbb{N}$ , define  $\theta_N$  to be the first stopping time such that

$$L(x, \theta_N) = \sum_{k=0}^{\theta_N} \mathbb{I}\{X_k > x\} = N.$$

Hence,

$$\begin{aligned}
\mathbb{E}_y Y_{T(t)} &\geq \mathbb{E}_y Y_{\theta_N \wedge T(t)} + \mathbb{E}_y \sum_{n=\theta_N+1}^{T(t)} \mathbb{I}\{X_n > x\} \\
&= \mathbb{E}_y Y_{\theta_N \wedge T(t)} + \mathbb{E}_y \{L(x, T(t)) - N; L(x, T(t)) > N\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_y\{L(x, T(t)) - N; L(x, T(t)) > N\} &\leq \mathbb{E}_y(Y_{T(t)} - Y_{\theta_N \wedge T(t)}) \\
&= \mathbb{E}_y(V_x(\tilde{X}_{T(t)}) - V_x(\tilde{X}_{\theta_N \wedge T(t)})) \\
&\leq \mathbb{E}_y\{V(\tilde{X}_{T(t)}) - V(x); T(t) > \theta_N\}.
\end{aligned}$$

Since  $\tilde{X}_{T(t)} \leq t + s(t)$ ,

$$\begin{aligned}
\mathbb{E}_y\{L(x, T(t)) - N; L(x, T(t)) > N\} \\
\leq (V(t + s(t)) - V(x))\mathbb{P}_y\{L(x, T(t)) > N\}. \quad (3.21)
\end{aligned}$$

Taking

$$N := [A(V(t + s(t)) - V(x))] + 1,$$

we get from (3.21) that the mean in (3.20) isn't greater than

$$\mathbb{P}_y\{L(x, T(t)) > N\},$$

which in its turn is not greater than

$$\frac{\mathbb{E}_y L(x, T(t))}{N},$$

by the Markov inequality. Due to the upper bound provided by Lemma 3.2,

$$\frac{\mathbb{E}_y L(x, T(t))}{N} \leq \frac{1}{A} \rightarrow 0 \quad \text{as } A \rightarrow \infty,$$

and the proof of uniform integrability of (3.16) is complete.  $\square$

### 3.2 Integro-local upper bound for renewal function

A transient Markov chain  $X_n$  visits any bounded set finitely many times at the most. In the next result we describe the asymptotic behaviour of the renewal functions

$$\begin{aligned}
H_y(x, x+h] &:= \sum_{n=0}^{\infty} \mathbb{P}_y\{x < X_n \leq x+h\}, \\
H(x, x+h] &:= \sum_{n=0}^{\infty} \mathbb{P}\{x < X_n \leq x+h\} = \int_0^{\infty} H_y(x, x+h] \mathbb{P}\{X_0 \in dy\}.
\end{aligned}$$

**Theorem 3.3.** *Let the condition (3.6) hold for some  $r(x)$  satisfying (3.1) and increasing function  $s(x) = o(1/r(x))$ . Assume (3.14) for some decreasing  $v(x)$  satisfying*

$$c_v := \sup_x \frac{v(x)}{v(2x)} < \infty. \quad (3.22)$$

Assume also

$$\mathbb{P}\{\xi(x) \leq -s(x)\} \leq p(x)v(x) \text{ for all } x \geq \widehat{x}, \quad (3.23)$$

where a decreasing function  $p(x) > 0$  is integrable. Then the family of random variables

$$v(x)r(x) \sum_{n=0}^{\infty} \mathbb{I}\{x < X_n \leq x + 1/r(x)\}, \quad x > x_*, \quad X_0 = y,$$

is uniformly integrable for some  $x_* > \widehat{x}$ .

In particular, there exists a  $c_1 < \infty$  such that

$$H_y(x, x + 1/r(x)) \leq \frac{c_1}{v(x)r(x)} \text{ for all } x > x_* \text{ and } y,$$

and further, for some  $c_2 < \infty$ ,

$$H_y(x_*, x] \leq c_2 \frac{x+1}{v(x)} \text{ for all } x > x_* \text{ and } y.$$

*Proof.* Clearly, (3.23) is stronger than (2.105). This implies the transience of  $X_n$ .

It follows from (3.22) that, for any natural  $A$ ,

$$\frac{v(x)}{v(x+Ax)} \leq (c_v)^A \text{ for all } x. \quad (3.24)$$

Considering the first enter of  $X_n$  into the segment  $(x, x + 1/r(x)]$  we see that the theorem conclusion is equivalent to the uniform integrability of the family

$$v(x)r(x) \sum_{n=0}^{\infty} \mathbb{I}\{x < X_n \leq x + 1/r(x)\}, \quad x > x_*, \quad X_0 = y, \quad y \in (x, x + 1/r(x)]. \quad (3.25)$$

First consider the Markov chain  $Y_n$  with jumps

$$\eta(x) := \xi(x) \mathbb{I}\{\xi(x) > -s(x)\}.$$

This Markov chain satisfies the condition (3.7) because  $\eta(x) > -s(x)$ , and the condition (3.6). So Lemma 3.1 is applicable to the chain  $Y_n$ . Then there exist  $\delta \in (0, \varepsilon)$  and  $x_*$  such that

$$\mathbb{P}\{Y_n \leq x \text{ for some } n \geq 1 \mid Y_0 = y\} \leq e^{\delta(R(x)-R(y))} \text{ for all } y > x > x_*. \quad (3.26)$$

Fix a natural  $A > 1$  and consider the stopping time

$$T^Y(t) = \min\{n \geq 1 : Y_n > t\} \text{ where } t := x + A/r(x).$$

For any  $Y_0 = y \in (x, x + 1/r(x)]$ ,

$$\begin{aligned} & v(x)r(x) \sum_{n=0}^{T^Y(t)-1} \mathbb{I}\{x < Y_n \leq x + 1/r(x)\} \\ & \leq \frac{v(x)}{v(t+s(t))} r(x)(t+s(t)-x) \frac{v(t+s(t))}{t+s(t)-x} \sum_{n=0}^{T^Y(t)-1} \mathbb{I}\{Y_n > x\}. \end{aligned}$$

By (3.24), for all  $x > \hat{x}$  such that  $s(x) \leq 1/r(x)$  and  $1/r(x) \leq x$ ,

$$\frac{v(x)}{v(t+s(t))} \leq \frac{v(x)}{v(x+(A+1)/r(x))} \leq (c_v)^{A+1}. \quad (3.27)$$

Further,

$$r(x)(t+s(t)-x) = r(x)(A/r(x)+s(t)) \rightarrow A \quad \text{as } x \rightarrow \infty. \quad (3.28)$$

Finally, the family with respect to  $x > \hat{x}$ ,  $Y_0 = y$ ,  $y \in (x, x+1/r(x)]$  of random variables

$$\frac{v(t+s(t))}{t+s(t)-x} \sum_{n=0}^{T^Y(t)-1} \mathbb{I}\{Y_n > x\}$$

is uniformly integrable, due to Lemma 3.2 applied to the chain  $Y_n$ . So, the family of random variables

$$v(x)r(x) \sum_{n=0}^{T^Y(t)-1} \mathbb{I}\{x < Y_n \leq x+1/r(x)\}$$

is uniformly integrable too.

Further, after the stopping time  $T^Y(x+A/r(x))$  the chain  $Y_n$  falls below the level  $x+1/r(x)$  with probability  $e^{\delta(R(x+1/r(x))-R(x+A/r(x)))}$  at the most, see (3.26). Since

$$e^{\delta(R(x+1/r(x))-R(x+A/r(x)))} \leq e^{-\delta(A-1)/(1+cA)}$$

provided  $A > 1$ —see (2.7), we obtain by the Markov property that

$$v(x)r(x) \sum_{n=0}^{\infty} \mathbb{I}\{x < Y_n \leq x+1/r(x)\}$$

is majorised by geometric number at the most of summands taken from the uniformly integrable family of random variables, which yields the lemma conclusion for the chain  $Y_n$ . In particular,

$$H_y^Y(x, x+1/r(x)] \leq \frac{c_3}{v(x)r(x)} \quad \text{for all } x > x_* \text{ and } y. \quad (3.29)$$

Further, in order to pass from  $Y_n$  to  $X_n$  we first notice the inequality, for any  $x < y$ ,

$$\begin{aligned} \mathbb{P}\{X_n \leq x \text{ for some } n \geq 1 \mid X_0 = y\} &\leq \mathbb{P}\{Y_n \leq x \text{ for some } n \geq 1 \mid Y_0 = y\} \\ &+ \mathbb{P}\{X_n \neq Y_n \text{ for some } n \geq 1, Y_k > x \text{ for all } k \geq 1 \mid Y_0 = y\}. \end{aligned}$$

For all  $x < y$  satisfying  $x - s(x) > \hat{x}$ , the second probability on the right is not greater than

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \mathbb{P}\{\xi(Y_n) < -s(Y_n), Y_n > x \mid Y_0 = y\} \\
 &= \sum_{n=0}^{\infty} \int_x^{\infty} \mathbb{P}\{\xi(z) < -s(z)\} \mathbb{P}\{Y_n \in dz \mid Y_0 = y\} \\
 &= \int_x^{\infty} \mathbb{P}\{\xi(z) < -s(z)\} H_y^Y(dz) \\
 &\leq \int_x^{\infty} p(z)v(z)H_y^Y(dz),
 \end{aligned}$$

by the condition (3.23). The latter integral goes to 0 as  $x \rightarrow \infty$ . Indeed, both functions  $p(z)$  and  $v(x)$  are decreasing, so that

$$\int_x^{\infty} p(z)v(z)H_y^Y(dz) \leq \sum_{i=0}^{\infty} p(x_i)v(x_i)H_y^Y(x_i, x_{i+1}],$$

where  $x_0 := x$  and  $x_{i+1} := x_i + 1/r(x_i)$ . Then, in view of the upper bound (3.29) for  $H_y^Y(z, z + 1/r(z)]$ ,

$$\int_x^{\infty} p(z)v(z)H_y^Y(dz) \leq c_3 \sum_{i=0}^{\infty} \frac{p(x_i)}{r(x_i)}.$$

By (2.7),

$$\begin{aligned}
 \sum_{i=0}^{\infty} \frac{p(x_i)}{r(x_i)} &= \sum_{i=0}^{\infty} \frac{p(x_i)}{r(x_{i-1} + 1/r(x_{i-1}))} \\
 &\leq c_4 \sum_{i=0}^{\infty} \frac{p(x_i)}{r(x_{i-1})} \\
 &= c_4 \sum_{i=0}^{\infty} p(x_i)(x_i - x_{i-1}).
 \end{aligned}$$

Then monotonicity of the function  $p(x)$  yields

$$\sum_{i=0}^{\infty} \frac{p(x_i)}{r(x_i)} \leq c_4 \int_x^{\infty} p(u)du \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

because  $p(x)$  is integrable. Hence,

$$\int_x^{\infty} p(z)v(z)H_y^Y(dz) \rightarrow 0 \quad \text{as } x \rightarrow \infty \text{ uniformly for } y > x. \quad (3.30)$$

Together with (3.26) it yields that

$$\mathbb{P}\{X_n \leq x \text{ for some } n \geq 1 \mid X_0 = y\} \leq e^{\delta(R(x)-R(y))} + o(1) \quad (3.31)$$

as  $x \rightarrow \infty$  uniformly for  $y > x$ .

In the same way as it was done for  $Y_n$ , we now fix a natural  $A > 1$  and consider the stopping time

$$T(t) = \min\{n \geq 1 : X_n > t\} \quad \text{where } t := x + A/r(x).$$

For any  $X_0 = y \in (x, x + 1/r(x)]$ , the family with respect to  $x > \hat{x}$ ,  $X_0 = y$ ,  $y \in (x, x + 1/r(x)]$  of random variables

$$v(x)r(x) \sum_{n=0}^{T(t)-1} \mathbb{I}\{x < X_n \leq x + 1/r(x)\}$$

is uniformly integrable, due to (3.27), (3.28) and Lemma 3.2 applied to  $X_n$ .

Further, after the stopping time  $T(x + A/r(x))$  the chain  $X_n$  falls below the level  $x + 1/r(x)$  with probability  $e^{\delta(R(x+1/r(x)) - R(x+A/r(x)))} + o(1)$  at the most, see (3.31). By the same reasons as for the Markov chain  $Y_n$ ,

$$v(x)r(x) \sum_{n=0}^{\infty} \mathbb{I}\{x < X_n \leq x + 1/r(x)\}$$

is majorised by geometric number at the most of summands taken from the uniformly integrable family of random variables, which yields the theorem conclusion for the chain  $X_n$ .  $\square$

### 3.3 Factorisation result for renewal function with weights

In this section, either  $n(x) \equiv \infty$  or  $n(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Let  $A(x) \subset \mathbb{R}^+$  be a family of Borel sets.

For a function  $q(z) \geq 0$  on  $\mathbb{R}^+$ , we look at impact of  $q(z)$  on the asymptotic behavior of the partial renewal measure with weights

$$\sum_{n=0}^{n(x)} \mathbb{E}\{e^{-\sum_{k=0}^{n-1} q(X_k)}; X_n \in A(x)\}, \quad (3.32)$$

compared to that of

$$\sum_{n=0}^{n(x)} \mathbb{P}\{X_n \in A(x)\}.$$

**Lemma 3.4.** *Let  $a(x) > 0$  be a function on  $\mathbb{R}^+$  and  $c > 0$  be a constant. Let the family of random variables*

$$a(x) \sum_{n=0}^{n(x)} \mathbb{I}\{X_n \in A(x)\}, \quad x > 0, \quad X_0 = z, \quad (3.33)$$



be uniformly integrable and let there exist a  $c > 0$  such that, for all  $N \in \mathbb{Z}^+$  and  $z \geq 0$ ,

$$a(x) \sum_{n=N}^{n(x)} \mathbb{P}_z\{X_n \in A(x)\} \rightarrow c \quad \text{as } x \rightarrow \infty. \quad (3.34)$$

If  $q(z) \geq 0$ , then

$$a(x) \sum_{n=0}^{n(x)} \mathbb{E}\{e^{-\sum_{k=0}^{n-1} q(X_k)}; X_n \in A(x)\} \rightarrow c \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \quad \text{as } x \rightarrow \infty.$$

*Proof.* The condition (3.34) particularly yields, for any fixed  $N \in \mathbb{N}$ ,

$$a(x) \sum_{n=0}^{N-1} \mathbb{P}\{X_n \in A(x)\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Then

$$\begin{aligned} & \left| a(x) \sum_{n=0}^{n(x)} \mathbb{E}\{e^{-\sum_{k=0}^{n-1} q(X_k)}; X_n \in A(x)\} - c \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \right| \\ &= a(x) \left| \sum_{n=0}^{n(x)} \mathbb{E}\{e^{-\sum_{k=0}^{n-1} q(X_k)}; X_n \in A(x)\} - \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \sum_{n=0}^{n(x)} \mathbb{P}\{X_n \in A(x)\} \right| + o(1) \\ &= a(x) \left| \mathbb{E} \sum_{n=N}^{n(x)} \left( e^{-\sum_{k=0}^{n-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \right) \mathbb{I}\{X_n \in A(x)\} \right| + o(1). \end{aligned}$$

In its turn, the absolute value of the mean on the right is not greater than the sum of the absolute values of the mean values of the following random variables:

$$\begin{aligned} \zeta_1(x, N) &:= \sum_{n=N}^{n(x)} \left( e^{-\sum_{k=0}^{n-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{n-1} q(X_k)} \right) \mathbb{I}\{X_n \in A(x)\}, \\ \zeta_2(x, N) &:= \sum_{n=N}^{n(x)} \left( e^{-\sum_{k=0}^{n-1} q(X_k)} - e^{-\sum_{k=0}^{N-1} q(X_k)} \right) \mathbb{I}\{X_n \in A(x)\}, \\ \zeta_3(x, N) &:= \sum_{n=N}^{n(x)} \left( \mathbb{E}e^{-\sum_{k=0}^{n-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \right) \mathbb{I}\{X_n \in A(x)\}. \end{aligned}$$

By the condition (3.33), both families of random variables  $\{a(x)\zeta_2(x, N), x > 0, N \geq 1\}$  and  $\{a(x)\zeta_3(x, N), x > 0, N \geq 1\}$  are uniformly integrable. Then, taking into account that  $q(z) \geq 0$  implies

$$e^{-\sum_{k=0}^{N-1} q(X_k)} \xrightarrow{a.s.} e^{-\sum_{k=0}^{\infty} q(X_k)} \quad \text{as } N \rightarrow \infty,$$

we conclude that both  $\sup_x a(x)E_2(x, N)$  and  $\sup_x a(x)E_3(x, N)$  go to 0 as  $N \rightarrow \infty$ . This proves the required result if we additionally show that, for any fixed  $N$ ,

$$a(x)\mathbb{E}\zeta_1(x, N) \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (3.35)$$

Indeed, conditioning on  $X_0, \dots, X_{N-1}$ , leads to the equality

$$\begin{aligned} & \mathbb{E}\zeta_1(x, N) \\ &= \mathbb{E}\left\{\left(e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{N-1} q(X_k)}\right)\mathbb{E}\left\{\sum_{n=N}^{n(x)} \mathbb{I}\{X_n \in A(x)\} \middle| X_0, \dots, X_{N-1}\right\}\right\} \\ &= \mathbb{E}\left(e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{N-1} q(X_k)}\right)\mathbb{E}_{X_{N-1}}\sum_{n=N}^{n(x)} \mathbb{I}\{X_n \in A(x)\}, \end{aligned}$$

by the Markov property. By the uniform integrability (3.33), the family of random variables

$$a(x)\left(e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{N-1} q(X_k)}\right)\mathbb{E}_{X_{N-1}}\sum_{n=N}^{n(x)} \mathbb{I}\{X_n \in A(x)\}, \quad x > 0,$$

is uniformly integrable too. By the condition (3.34),

$$a(x)\mathbb{E}_{X_{N-1}}\sum_{n=N}^{n(x)} \mathbb{I}\{X_n \in A(x)\} \xrightarrow{a.s.} c \quad \text{as } x \rightarrow \infty.$$

This allows us to conclude that, as  $x \rightarrow \infty$ ,

$$a(x)\mathbb{E}\zeta_1(x, N) \rightarrow c\mathbb{E}\left(e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{N-1} q(X_k)}\right) = 0,$$

and (3.35) follows which completes the proof.  $\square$

**Lemma 3.5.** *Let  $p$  be a number between 0 and 1 and  $A_n$  be a sequence of Borel sets from  $\mathbb{R}^+$  such that, for any  $z \geq 0$ ,*

$$\mathbb{P}_z\{X_n \in A_n\} \rightarrow p \quad \text{as } n \rightarrow \infty. \quad (3.36)$$

*If  $q(z) \geq 0$ , then*

$$\mathbb{E}e^{-\sum_{k=0}^{n-1} q(X_k)}\mathbb{I}\{X_n \in A_n\} \rightarrow p\mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \quad \text{as } n \rightarrow \infty.$$

*Proof.* Fix any  $N \in \mathbb{N}$ . Then

$$\begin{aligned} & \left| \mathbb{E}e^{-\sum_{k=0}^{n-1} q(X_k)}\mathbb{I}\{X_n \in A_n\} - \mathbb{P}\{X_n \in A_n\}\mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \right| \\ & \leq \left| \mathbb{E}\left(e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{N-1} q(X_k)}\right)\mathbb{I}\{X_n \in A_n\} \right| \\ & \quad + \left| \mathbb{E}\left(e^{-\sum_{k=0}^{n-1} q(X_k)} - e^{-\sum_{k=0}^{N-1} q(X_k)}\right) \right| \\ & \quad + \left| \mathbb{E}e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)} \right| \\ & =: |E_1(N, n, A)| + |E_2(N, n)| + |E_3(N)|. \end{aligned}$$

We have  $E_2(N, n) \rightarrow 0$  and  $E_3(N) \rightarrow 0$  as  $n, N \rightarrow \infty$  because  $q(z) \geq 0$ . Further, conditioning on  $X_0, \dots, X_{N-1}$ , leads to the equality

$$\begin{aligned} E_1(N, n, A) &= \mathbb{E} \left\{ \left( e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E} e^{-\sum_{k=0}^{N-1} q(X_k)} \right) \mathbb{P}\{X_n \in A \mid X_0, \dots, X_{N-1}\} \right\} \\ &= \mathbb{E} \left( e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E} e^{-\sum_{k=0}^{N-1} q(X_k)} \right) \mathbb{P}\{X_n \in A \mid X_N\}, \end{aligned}$$

by the Markov property. By the condition (3.36),

$$\mathbb{P}\{X_n \in A \mid X_N\} \xrightarrow{a.s.} p \quad \text{as } n \rightarrow \infty.$$

This allows us to conclude that, for any fixed  $N$ , as  $n \rightarrow \infty$ ,

$$E_1(N, n, A) \rightarrow p \mathbb{E} \left( e^{-\sum_{k=0}^{N-1} q(X_k)} - \mathbb{E} e^{-\sum_{k=0}^{N-1} q(X_k)} \right) = 0,$$

and the proof is complete.  $\square$

### 3.4 Thresholds

Consider a new Markov chain  $Z_n$ ,  $n \geq 0$ , whose jumps  $\zeta(x)$  are just truncations of the original jumps  $\xi(x)$  at levels  $-s(x)$  and  $s(x)$ , that is,

$$\zeta(x) = \xi(x) \mathbb{I}\{|\xi(x)| \leq s(x)\}.$$

**Lemma 3.6.** *Let  $X_n \rightarrow \infty$  with probability 1, the condition (3.6) hold for some  $r(x)$  satisfying (3.1) and increasing function  $s(x) = o(1/r(x))$ . Let also the condition (3.14) hold for some  $v(x)$  satisfying (3.22), and*

$$\mathbb{P}\{|\xi(x)| > s(x)\} \leq p(x)v(x), \quad (3.37)$$

for all  $x$ , where a decreasing function  $p(x) > 0$  is integrable. Then, for any  $\varepsilon > 0$  there exists an  $x_\varepsilon$  such that

$$\mathbb{P}\{Z_n \neq X_n \text{ for some } n \geq 0\} \leq \varepsilon$$

provided  $Z_0 = X_0 > x_\varepsilon$ .

*Proof.* For any  $z$ ,

$$\begin{aligned} &\mathbb{P}\{Z_n \neq X_n \text{ for some } n \mid X_0 = y\} \\ &\leq \mathbb{P}\{X_n \leq z \text{ for some } n \mid X_0 = y\} \\ &\quad + \mathbb{P}\{Z_n \neq X_n \text{ for some } n, X_n \geq z \text{ for all } n \mid X_0 = y\}. \end{aligned}$$

Since  $X_n \xrightarrow{a.s.} \infty$ , there is an  $x_1(z)$  such that

$$\mathbb{P}\{X_n \leq z \text{ for some } n \mid X_0 = y\} \leq \varepsilon/2 \quad \text{for all } y > x_1(z).$$

Given  $Z_0 = X_0 > z$ ,

$$\begin{aligned}
& \mathbb{P}\{Z_n \neq X_n \text{ for some } n, X_n > z \text{ for all } n \mid X_0 = y\} \\
& \leq \sum_{n=0}^{\infty} \mathbb{P}\{|\xi(X_n)| > s(X_n), X_n > z \mid X_0 = y\} \\
& = \int_z^{\infty} \mathbb{P}\{|\xi(x)| > s(x)\} H_y^X(dx) \\
& \leq 2 \int_z^{\infty} p(x)v(x) H_y^X(dx).
\end{aligned}$$

An integro-local upper bound for  $H_y^X$  is delivered in Theorem 3.3. Then the integral on the right is less than  $\varepsilon/2$  for sufficiently large  $z = z(\varepsilon)$ , by the same reasons as (3.30) holds. This concludes the proof with  $x_\varepsilon = x_1(z(\varepsilon))$ .  $\square$

### 3.5 Convergence to $\Gamma$ -distribution for transient chains

In this section we are interested in the asymptotic growth rate of a Markov chain  $X_n$  that goes to infinity with probability 1 as  $n \rightarrow \infty$ . It happens if this chain is transient. First time a limit theorem for Markov chain with asymptotically zero drift was produced by Lamperti in [49] where the convergence to  $\Gamma$ -distribution was proven for the case of jumps with all moments finite. The proof is based on the method of moments. The results from Lamperti [49] have been generalised by Klebaner [42] and later by Kersting [40]. The author of the latter paper works under the assumption that the moments of order  $2 + \delta$  are bounded.

**Theorem 3.7.** *Suppose there exist  $b > 0$  and  $\mu > b/2$  such that, for some increasing function  $s(x) = o(x)$ ,*

$$m_1^{[s(x)]}(x) \sim \mu/x \quad \text{and} \quad m_2^{[s(x)]}(x) \rightarrow b \quad \text{as } x \rightarrow \infty, \quad (3.38)$$

and

$$\mathbb{P}\{|\xi(x)| > s(x)\} \leq p(x)/x \quad \text{for all } x, \quad (3.39)$$

where a decreasing function  $p(x) > 0$  is integrable. If

$$\limsup_{n \rightarrow \infty} X_n = \infty \quad \text{with probability 1,}$$

then  $X_n^2/n$  converges weakly to  $\Gamma$ -distribution with mean  $2\mu + b$  and variance  $(2\mu + b)2b$ , that is,  $\Gamma_{1/2+\mu/b, 2b}$ .

Let us give a sufficient condition for (3.39). If the family  $|\xi(x)|$ ,  $x \geq 0$ , possesses a square integrable majorant  $\Xi$ ,  $\mathbb{E}\Xi^2 < \infty$ , that is,  $|\xi(x)| \leq_{st} \Xi$  for all  $x$ , then there exists an increasing function  $s(x) = o(x)$  such that (3.39) holds.

*Proof.* Since (3.38) holds with  $\mu > b/2$ ,  $X_n$  satisfies the condition (2.104) with sufficiently large  $\hat{x}$  and  $r(x) = \gamma/x$  for some  $\gamma \in (1, 2\mu/b)$ . Moreover, (3.39) implies that (2.105) holds. Therefore,  $X_n$  is transient by Theorem 2.18.

The chain  $X_n$  also satisfies (3.14) with  $v(x) = \mu/2x$  because

$$\begin{aligned} \mathbb{E}\{\xi(x); \xi(x) \leq s(x)\} &= m_1^{[s(x)]}(x) + \mathbb{E}\{\xi(x); \xi(x) \leq -s(x)\} \\ &\geq m_1^{[s(x)]}(x) - x\mathbb{P}\{\xi(x) \leq -s(x)\} \\ &\geq m_1^{[s(x)]}(x) - p(x), \end{aligned} \quad (3.40)$$

by the condition (3.39), and  $p(x) = o(1/x)$  as decreasing integrable function.

Given any  $y_0 > 0$ , it is sufficient to prove convergence to  $\Gamma$ -distribution for  $X_0 > y_0$  only because  $X_n \rightarrow \infty$  with probability 1.

The conditions (3.6), (3.14) and (3.39) allow to apply Lemma 3.6 to the chain  $X_n$ . Hence it suffices to prove the same result for the chain  $Z_n$ , that is, it is sufficient to prove that,

$$\frac{Z_n^2}{n} \Rightarrow \Gamma_{(2\mu+b)/2b, 2b}. \quad (3.41)$$

Again by Lemma 3.6, for all sufficiently large  $Z_0 = X_0$ ,

$$\mathbb{P}\{Z_n \neq X_n \text{ for some } n \geq 0\} < 1,$$

and for those initial conditions—that are only considered in the sequel—the Markov property yields the convergence

$$Z_n \xrightarrow{a.s.} \infty \quad \text{as } n \rightarrow \infty. \quad (3.42)$$

For all  $x$ ,

$$\mathbb{E}\zeta(x) = m_1^{[s(x)]}(x) \quad \text{and} \quad \mathbb{E}\zeta^2(x) = m_2^{[s(x)]}(x). \quad (3.43)$$

In addition, the inequality  $|\zeta(x)| \leq s(x) = o(x)$  implies that, for every  $j \geq 3$ ,

$$|\mathbb{E}\zeta^j(x)| \leq m_2^{[s(x)]}(x)s^{j-2}(x) = o(x^{j-2}) \quad \text{as } x \rightarrow \infty. \quad (3.44)$$

Let us compute the mean of the increment of  $Z_n^{2i}$ . For  $i = 1$  we have

$$\begin{aligned} \mathbb{E}\{Z_{n+1}^2 - Z_n^2 | Z_n = x\} &= \mathbb{E}(2x\zeta(x) + \zeta^2(x)) \\ &= 2\mu + b + o(1) \end{aligned}$$

as  $x \rightarrow \infty$ , by (3.43). Applying now (3.42) we get

$$\mathbb{E}(Z_{n+1}^2 - Z_n^2) \rightarrow 2\mu + b \quad \text{as } n \rightarrow \infty.$$

Hence,

$$\mathbb{E}Z_n^2 \sim (2\mu + b)n \quad \text{as } n \rightarrow \infty. \quad (3.45)$$

Let now  $i \geq 2$ . We have

$$\begin{aligned}
& \mathbb{E}\{Z_{n+1}^{2i} - Z_n^{2i} | Z_n = x\} \\
&= \mathbb{E}\left(2ix^{2i-1}\zeta(x) + i(2i-1)x^{2i-2}\zeta^2(x) + \sum_{l=3}^{2i} x^{2i-l}\zeta^l(x) \binom{2i}{l}\right) \\
&= i[2\mu + (2i-1)b + o(1)]x^{2i-2} + \sum_{l=3}^{2i} x^{2i-l}\mathbb{E}\zeta^l(x) \binom{2i}{l} \quad (3.46)
\end{aligned}$$

as  $x \rightarrow \infty$ , by (3.43). Owing to (3.44),

$$\sum_{l=3}^{2i} x^{2i-l}\mathbb{E}\zeta^l(x) \binom{2i}{l} = \sum_{l=3}^{2i} x^{2i-l}o(x^{l-2}) = o(x^{2i-2}) \quad \text{as } x \rightarrow \infty.$$

Substituting this into (3.46) with  $x = Z_n$  and taking into account (3.42), we deduce that

$$\mathbb{E}\{Z_{n+1}^{2i} - Z_n^{2i}\} = i[2\mu + (2i-1)b + o(1)]\mathbb{E}Z_n^{2i-2}. \quad (3.47)$$

In particular, for  $i = 2$  we get

$$\begin{aligned}
\mathbb{E}\{Z_{n+1}^4 - Z_n^4\} &= 2(2\mu + 3b)\mathbb{E}Z_n^2 + o(n) \\
&\sim 2(2\mu + 3b)(2\mu + b)n,
\end{aligned}$$

due to (3.45). It implies that

$$\mathbb{E}Z_n^4 \sim (2\mu + 3b)(2\mu + b)n^2 \quad \text{as } n \rightarrow \infty.$$

By induction arguments, we deduce from (3.47) that, as  $n \rightarrow \infty$ ,

$$\mathbb{E}Z_n^{2i} \sim n^i \prod_{k=1}^i (2\mu + (2k-1)b),$$

which yields (3.41), that is,  $Z_n^2/n$  weakly converges to Gamma distribution with mean  $2\mu + b$  and variance  $2b(2\mu + b)$  and the proof is complete.  $\square$

### 3.6 Integral renewal theorem for transient chain with Gamma limit

If the Markov chain  $X_n$  is transient then it visits any bounded set finitely many times at the most. The next result determines the asymptotic behaviour of the renewal functions  $H_y(x)$  and  $H(x)$  in the case of convergence to  $\Gamma$ -distribution.

**Theorem 3.8.** *Let the conditions of Theorem 3.7 hold. Then, for any initial distribution of the chain  $X$  and for any fixed  $B > 0$ ,*

$$\sum_{n=0}^{[Bx^2]} \mathbb{P}\{X_n \in (x_*, x]\} \sim I(B)x^2 \quad \text{as } x \rightarrow \infty,$$

where

$$I(y) := \int_0^y \Gamma(1/z) dz = y\Gamma(1/y) + \int_{1/y}^\infty \frac{1}{z} \gamma(z) dz, \quad I(\infty) = \frac{1}{2\mu - b},$$

where  $x_*$  is defined in Theorem 3.3, and  $\Gamma(t)$  and  $\gamma(t)$  denote the cumulative distribution function and probability density function respectively of the  $\Gamma$ -distribution with mean  $2\mu + b$  and variance  $(2\mu + b)2b$ . In addition,

$$H(x_*, x] \sim \frac{1}{2\mu - b} x^2 \quad \text{as } x \rightarrow \infty.$$

*Proof.* By Theorem 3.7, for every fixed  $B$ ,

$$\begin{aligned} \sum_{n=0}^{[Bx^2]} \mathbb{P}\{X_n \in (x_*, x]\} &= \sum_{n=0}^{[Bx^2]} (\Gamma(x^2/n) + o(1)) \\ &= \sum_{n=0}^{[Bx^2]} \Gamma(x^2/n) + o(x^2) \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Due to

$$\sum_{n=0}^{[Bx^2]} \Gamma(x^2/n) \sim x^2 \int_0^B \Gamma(1/z) dz \quad \text{as } x \rightarrow \infty,$$

the first conclusion follows. Further, since

$$\int_0^B \Gamma(1/z) dz \rightarrow \frac{1}{2\mu - b} \quad \text{as } B \rightarrow \infty,$$

we conclude the lower bound

$$\liminf_{x \rightarrow \infty} \frac{H(x_*, x]}{x^2} \geq \frac{1}{2\mu - b}. \quad (3.48)$$

For arbitrary  $y$ , let us now prove the upper bound

$$\limsup_{x \rightarrow \infty} \frac{H_y(x_*, x]}{x^2} \leq \frac{1}{2\mu - b}. \quad (3.49)$$

For any  $A > 1$ , by (3.31) and the Markov property,

$$\begin{aligned} H_y(x_*, x] &\leq \mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \in (x_*, x]\} \\ &\quad + (e^{\delta(R(x)-R(Ax))} + o(1)) \sup_{z \leq x} H_z(x_*, x] \end{aligned} \quad (3.50)$$

as  $x \rightarrow \infty$  uniformly for all  $A > 1$ . Since  $r(x) = \gamma/x$ ,

$$e^{\delta(R(x)-R(Ax))} = 1/A^{\delta\gamma}.$$

Thus, applying the upper bound proven in Theorem 3.3 on the right side of (3.50) we deduce that, for some  $c < \infty$ ,

$$H_y(x_*, x] \leq \mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \in (x_*, x]\} + (c/A^{\delta\gamma} + o(1))x^2 \quad (3.51)$$

as  $x \rightarrow \infty$  uniformly for all  $A > 1$ . The expectation of the sum on the right hand side of (3.51) may be estimated as follows: for  $C > 1$ ,

$$\begin{aligned} \mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \in (x_*, x]\} &\leq \mathbb{E}_y \sum_{n=0}^{[CA^2x^2]} \mathbb{I}\{X_n \in (x_*, x]\} \\ &\quad + \mathbb{E}_y \left\{ \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n > x_*\}; T(Ax) > CA^2x^2 \right\}. \end{aligned}$$

The second term on the right side is not greater than

$$\begin{aligned} &\mathbb{E}_y \left\{ L(x_*, T(Ax)); X_n \leq x_* \text{ for some } n \geq A^2x^2 \right\} \\ &\quad + \mathbb{E}_y \left\{ L(x_*, T(Ax)); X_n > x_* \text{ for all } n \in [A^2x^2, T(Ax) - 1], T(Ax) > CA^2x^2 \right\} \\ &\leq \mathbb{E}_y \left\{ L(x_*, T(Ax)); X_n \leq x_* \text{ for some } n \geq A^2x^2 \right\} \\ &\quad + \mathbb{E}_y \left\{ L(x_*, T(Ax)); L(x_*, T(Ax)) > (C-1)A^2x^2 \right\}. \end{aligned}$$

By Lemma 3.2, the family of random variables

$$\frac{L(x_*, T(Ax))}{(Ax)^2}$$

is uniformly integrable, so, for any fixed  $A$ ,

$$\sup_{x > x_*, y} \frac{1}{x^2} \mathbb{E}_y \left\{ L(x_*, T(Ax)); L(x_*, T(Ax)) > (C-1)A^2x^2 \right\} \leq \psi(C),$$

where  $\psi(C) \rightarrow 0$  as  $C \rightarrow \infty$ . Since  $X_n \rightarrow \infty$  with probability 1,

$$\mathbb{P}\{X_n \leq x_* \text{ for some } n \geq A^2x^2\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Therefore, again by uniform integrability,

$$\frac{1}{x^2} \mathbb{E}_y \left\{ L(x_*, T(Ax)); X_n \leq x_* \text{ for some } n \geq A^2x^2 \right\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Altogether yields

$$\limsup_{x \rightarrow \infty} \sup_y \frac{1}{x^2} \mathbb{E}_y \left\{ \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n > x_*\}; T(Ax) > CA^2x^2 \right\} \leq \psi(C),$$



hence, uniformly for all  $y$ ,

$$\limsup_{x \rightarrow \infty} \frac{1}{x^2} \mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \in (x_*, x]\} \leq \mathbb{E}_y \sum_{n=0}^{[CA^2x^2]} \mathbb{I}\{X_n \in (x_*, x]\} + \psi(C),$$

which being substituted in (3.51) gives

$$\limsup_{x \rightarrow \infty} \frac{H_y(x_*, x]}{x^2} \leq \limsup_{x \rightarrow \infty} \frac{1}{x^2} \mathbb{E}_y \sum_{n=0}^{[CA^2x^2]} \mathbb{I}\{X_n \in (x_*, x]\} + \psi(C) + c/A^{\delta\gamma}$$

As already shown, as  $x \rightarrow \infty$ ,

$$\frac{1}{x^2} \sum_{n=0}^{[CA^2x^2]} \mathbb{P}_y\{X_n \in (x_*, x]\} \rightarrow I(CA^2),$$

which implies the following upper bound, for each fixed  $A, C > 1$ ,

$$\limsup_{x \rightarrow \infty} \frac{H_y(x_*, x]}{x^2} \leq I(CA^2) + \psi(C) + c/A^{\delta\gamma}.$$

Letting now first  $C \rightarrow \infty$  and then  $A \rightarrow \infty$ , we get the required upper bound (3.49). The lower (3.48) and upper (3.49) bounds yield the equivalence, for every fixed  $y$ ,

$$H_y(x_*, x] \sim \frac{1}{2\mu - b} x^2 \quad \text{as } x \rightarrow \infty.$$

Together with the uniform in  $y$  estimate of Theorem 3.3 this completes the proof.  $\square$

**Theorem 3.9.** *Let  $q(z) \geq 0$  and conditions of Theorem 3.7 hold. Then, for any initial distribution of the chain  $X$  and any fixed  $B \in (0, \infty]$*

$$\sum_{n=0}^{[Bx^2]} \mathbb{E}\{e^{-\sum_{k=0}^{n-1} q(X_k)}; X_n \in (x_*, x]\} \sim I(B)x^2 \mathbb{E}e^{-\sum_{k=0}^{\infty} q(X_k)}$$

as  $x \rightarrow \infty$ , where  $I(B)$  is defined in Theorem 3.8.

*Proof.* We may apply Lemma 3.4 because its condition (3.33) is guaranteed by Theorem 3.3, while the condition (3.34) by Theorem 3.8.  $\square$

### 3.7 The case $x\mathbb{E}\xi(x) \rightarrow \infty$ , LLN

In this section, let  $v(x) > 0$  be a decreasing differentiable function such that  $1/v(x)$  is concave,  $(1/v(x))' \downarrow 0$  as  $x \rightarrow \infty$ , so, by l'Hôpital's rule,  $xv(x) = x/(1/v(x)) \rightarrow \infty$  and  $x^2/V(x) \rightarrow \infty$  as  $x \rightarrow \infty$  where

$$V(x) := \int_0^x \frac{1}{v(y)} dy.$$

Since  $1/v$  is concave,

$$\frac{x}{2v(x)} \leq V(x) \leq \frac{x}{v(x)}, \quad v(\alpha x) \leq \frac{v(x)}{\alpha} \text{ for } \alpha \in [0, 1], \quad \text{and} \quad \left(\frac{1}{v(x)}\right)' \leq \frac{1}{v(x)x}, \quad (3.52)$$

and the inequality in the middle yields that, for any increasing function  $s(x) = o(x)$ ,

$$v(x \pm s(x)) \sim v(x). \quad (3.53)$$

Also assume that

$$xv(x) \text{ is increasing.} \quad (3.54)$$

**Theorem 3.10.** *Let, for some increasing function  $s(x) = o(x)$  as  $x \rightarrow \infty$ ,*

$$m_1^{[s(x)]}(x) \sim v(x), \quad (3.55)$$

$$m_2^{[s(x)]}(x) = o(xv(x)). \quad (3.56)$$

Let

$$\mathbb{P}\{|\xi(x)| > s(x)\} \leq p(x)v(x), \quad (3.57)$$

where  $p(x)$  is an integrable decreasing function. Assume also that

$$\limsup_{n \rightarrow \infty} X_n = \infty \quad \text{with probability 1.}$$

Then

$$\frac{V(X_n)}{n} \xrightarrow{p} 1 \quad \text{as } n \rightarrow \infty.$$

Let us give a sufficient condition for (3.56) and (3.57). If the family  $|\xi(x)|$ ,  $x \geq 0$ , possesses a majorant  $\Xi$  satisfying  $\mathbb{E}V(\Xi) < \infty$ , that is,  $|\xi(x)| \leq_{st} \Xi$  for all  $x$ , then there exists a function  $s(x) = o(x)$  such that (3.56) and (3.57) hold.

*Proof.* By the conditions (3.55) and (3.56),

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \geq 2/x \quad \text{for all sufficiently large } x,$$

so that the condition (3.6) holds with  $r(x) = 2/x$  and some  $\hat{x}$ , so hence the condition (2.104) holds and Theorem 2.18 implies that  $X_n \rightarrow \infty$  with probability 1 as  $n \rightarrow \infty$ .

As in (3.40), the conditions (3.55) and (3.57) imply that

$$\mathbb{E}\{\xi(x); \xi(x) \leq s(x)\} \sim v(x).$$

Then  $X_n$  satisfies the condition (3.14) with  $v(x)/2$  instead of  $v(x)$ . Therefore, Lemma 3.6 is applicable and then it suffices to prove that

$$\frac{V(Z_n)}{n} \xrightarrow{p} 1 \quad \text{as } n \rightarrow \infty, \quad (3.58)$$

where  $Z_n$  was introduced in Section 3.4.

Let us evaluate the expectation of the increment of  $V^{1+\alpha}(Z_n)$ ,  $\alpha \geq 0$ : for sufficiently large  $x$ ,

$$\begin{aligned}
& \mathbb{E}\{V^{1+\alpha}(Z_{n+1}) - V^{1+\alpha}(Z_n) | Z_n = x\} \\
&= \mathbb{E}\{V^{1+\alpha}(x + \xi(x)) - V^{1+\alpha}(x); |\xi(x)| \leq s(x)\} \\
&= (V^{1+\alpha})'(x)m_1^{[s(x)]}(x) + \mathbb{E}(V^{1+\alpha})''(x + \theta\xi(x))\xi^2(x)/2; |\xi(x)| \leq s(x)\} \\
&= (1 + \alpha)V^\alpha(x)\frac{1}{v(x)}m_1^{[s(x)]}(x) \\
&\quad + (1 + \alpha)\mathbb{E}\left\{\left(\alpha V^{\alpha-1}\frac{1}{v^2} - V^\alpha\frac{v'}{v^2}\right)(x + \theta\xi(x))\xi^2(x)/2; |\xi(x)| \leq s(x)\right\}.
\end{aligned} \tag{3.59}$$

Owing the condition (3.55), the first term on the right side equals

$$(1 + \alpha)V^\alpha(x)\frac{1}{v(x)}m_1^{[s(x)]}(x) = (1 + \alpha + o(1))V^\alpha(x) \quad \text{as } x \rightarrow \infty.$$

By (3.52),

$$\begin{aligned}
& \alpha \frac{V^{\alpha-1}(x+y)}{v^2(x+y)} - V^\alpha(x+y)\frac{v'(x+y)}{v^2(x+y)} \\
& \leq V^\alpha(x+y)\left(\frac{\alpha}{V(x+y)v^2(x+y)} + \frac{1}{(x+y)v(x+y)}\right) \\
& \leq (2\alpha + 1)V^\alpha(x+y)\frac{1}{(x+y)v(x+y)} \\
& = O(V^\alpha(x)/xv(x))
\end{aligned}$$

as  $x \rightarrow \infty$  uniformly on the set  $|y| \leq s(x) = o(x)$ , due to (3.53). Together with (3.56) this implies that the second term on the right side of (3.59) is of order  $o(V^\alpha(x))$  as  $x \rightarrow \infty$ . Substituting altogether into (3.59) we finally deduce that, as  $x \rightarrow \infty$ ,

$$\mathbb{E}\{V^{1+\alpha}(Z_{n+1}) - V^{1+\alpha}(Z_n) | Z_n = x\} = (1 + \alpha + o(1))V^\alpha(x). \tag{3.60}$$

Putting now  $\alpha = 0$  we get

$$\mathbb{E}\{V(Z_{n+1}) - V(Z_n) | Z_n = x\} \rightarrow 1 \quad \text{as } x \rightarrow \infty. \tag{3.61}$$

Applying here the convergence of  $Z_n \rightarrow \infty$ , see (3.42), we conclude that

$$\mathbb{E}\frac{V(Z_n)}{n} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \tag{3.62}$$

Next take  $\alpha = 1$ . Then

$$\mathbb{E}\{V^2(Z_{n+1}) - V^2(Z_n)\} = (2 + o(1))\mathbb{E}V(Z_n) \sim 2n \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\mathbb{E}\left(\frac{V(Z_n)}{n}\right)^2 \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Together with (3.62) it yields convergence of variance

$$\mathbb{V}ar \frac{V(Z_n)}{n} \rightarrow 0$$

which implies the desired convergence (3.58).  $\square$

### 3.8 The case $x\mathbb{E}\xi(x) \rightarrow \infty$ , SLLN

As usual, the strong law of large numbers requires more assumptions than the law of large numbers.

**Theorem 3.11.** *Let conditions of Theorem 3.10 hold. In addition, let*

$$m_2^{[s(x)]}(x) \leq \frac{xv(x)}{f(V(x))}, \quad (3.63)$$

for some increasing function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that both functions  $f(x)$  and  $x/f(x)$  are concave and

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)} < \infty. \quad (3.64)$$

Then

$$\frac{V(X_n)}{n} \xrightarrow{a.s.} 1 \quad \text{as } n \rightarrow \infty.$$

**Corollary 3.12.** *Let  $\mathbb{E}\xi(x) \sim c/x^\beta$ ,  $\beta \in [0, 1)$ , and*

$$\sup_x \mathbb{E}|\xi(x)|^{1+\beta} \log^{1+\delta}(1 + |\xi(x)|) < \infty,$$

for some  $\delta > 0$ . Then  $X_n^{1+\beta}/n \xrightarrow{a.s.} c(1+\beta)$  as  $n \rightarrow \infty$ .

**Corollary 3.13.** *Let  $\mathbb{E}\xi(x) \sim c(\log x)^{1+\beta}/x$ ,  $\beta > 0$ , and*

$$\sup_x \mathbb{E}\xi^2(x) < \infty.$$

Then  $X_n^2(\log X_n)^{-1-\beta}/n \xrightarrow{a.s.} 2c$  as  $n \rightarrow \infty$ .

Note that drifts like  $(\log x)/x$  or more speedy decreasing are excluded from consideration in Theorem 3.11 by the condition (3.64), so we cannot state the SLLN in such cases while Theorem 3.10 still provides the LLN.

*Proof of Theorem 3.11.* As in Theorem 3.10, it suffices to prove that

$$\frac{V(Z_n)}{n} \rightarrow 1 \quad \text{a.s. as } n \rightarrow \infty. \quad (3.65)$$

As was calculated in the previous proof with  $\alpha = 0$ ,

$$m(x) := \mathbb{E}\{V(Z_{n+1}) - V(Z_n) | Z_n = x\} \rightarrow 1 \quad \text{as } x \rightarrow \infty. \quad (3.66)$$

Put

$$\Delta_n = V(Z_{n+1}) - V(Z_n) - m(Z_n),$$

so that

$$V(Z_n) - V(Z_0) = \sum_{k=0}^{n-1} m(Z_k) + \sum_{k=0}^{n-1} \Delta_k.$$

By (3.66) and transience we have

$$\frac{1}{n} \sum_{k=0}^{n-1} m(Z_k) \rightarrow 1 \quad \text{a.s. as } n \rightarrow \infty,$$

and consequently the required convergence (3.65) will follow if we prove that

$$\frac{1}{n} \sum_{k=0}^{n-1} \Delta_k \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty. \quad (3.67)$$

Since the sequence  $Y_n(A)$  is a Markov chain, by the definition of  $\Delta_n$  we have

$$\mathbb{E}\{\Delta_n \mid Z_0, \dots, Z_n\} = \mathbb{E}\{\Delta_n \mid Z_n\} = 0.$$

Therefore, the process  $\sum_{k=0}^{n-1} \Delta_k$  constitutes a martingale with respect to the filtration  $\sigma(Z_0, \dots, Z_{n-1})$ . Prove that the increments of this martingale satisfy

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\Delta_n^2}{n^2} < \infty. \quad (3.68)$$

By the construction of  $\Delta_n$  and due to  $s(x) = o(x)$ ,

$$\begin{aligned} \mathbb{E}\{\Delta_n^2 \mid Z_n = x\} &\leq \mathbb{E}\{[V(x + \xi(x)) - V(x)]^2; |\xi(x)| \leq s(x)\} \\ &\leq c_1 (V'(x))^2 \mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\} \\ &\leq c_1 \frac{1}{v^2(x)} \frac{xv(x)}{f(x)} \leq 2c_1 \frac{V(x)}{f(x)}, \end{aligned}$$

owing to (3.52). Since the function  $y/f(y)$  is concave,

$$\mathbb{E}\Delta_n^2 \leq 2c_1 \frac{\mathbb{E}V(Z_n)}{f(\mathbb{E}V(Z_n))} \leq 2c_1 \frac{2n}{f(n/2)},$$

for sufficiently large  $n$ , as follows from (3.62). Then it follows from concavity of  $f(y)$  that  $\mathbb{E}\Delta_n^2 \leq 8c_1 n/f(n)$  which yields

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\Delta_n^2}{n^2} \leq 8c_1 \sum_{n=1}^{\infty} \frac{1}{nf(n)} < \infty,$$

by the condition (3.64), so that (3.68) holds and the proof is complete.  $\square$

### 3.9 Integro-local central limit theorem

In this section we continue to study the case where  $xm_1(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Since then the law of large numbers holds, it is natural to expect normal approximation of fluctuations around the mean value. In this section we deduce a version of CLT for a Markov chain starting at high level.

**Theorem 3.14.** *Let  $X_n \rightarrow \infty$  with probability 1. Let, for some increasing function  $s(x) = o(\sqrt{x/v(x)})$  where a decreasing function  $v(x)$  satisfies  $v(x/2) \leq c_v v(x)$ , (3.53) and  $xv(x) \rightarrow \infty$ ,*

$$m_1^{[s(x)]}(x) \sim v(x) \quad \text{and} \quad m_2^{[s(x)]}(x) \rightarrow b > 0 \quad \text{as } x \rightarrow \infty, \quad (3.69)$$

$$\mathbb{P}\{|\xi(x)| > s(x)\} \leq p(x)v(x) \quad (3.70)$$

and a decreasing function  $p(x) > 0$  is integrable. Then, for any fixed  $t > 0$  and  $h \in \mathbb{R}$ , given  $X_0 = x$ ,

$$\mathbb{P}_x\left\{X_n - x \leq \frac{h}{v(x)}\right\} - \Phi\left(\frac{h - nv^2(x)}{\sqrt{nbv^2(x)}}\right) \rightarrow 0$$

as  $x, n \rightarrow \infty$  in such a way that  $n \leq t/v^2(x)$ ; hereinafter  $\Phi$  stands for the standard normal distribution function. Moreover,

$$\sup_{x \leq y \leq x + o(1/v(x))} \left| \mathbb{P}_y\left\{X_n - x \leq \frac{h}{v(x)}\right\} - \Phi\left(\frac{h - nv^2(x)}{\sqrt{nbv^2(x)}}\right) \right| \rightarrow 0.$$

Notice that the condition (3.70) holds for some increasing function  $s(x) = o(\sqrt{x/v(x)})$  provided the family  $|\xi(x)|$ ,  $x \geq 0$ , possesses a square integrable majorant  $\Xi$ ,  $\mathbb{E}\Xi^2 < \infty$ , that is,  $|\xi(x)| \leq_{st} \Xi$  for all  $x$ .

We start with the following tightness result for  $X_n$ .

**Lemma 3.15.** *Let  $X_n \rightarrow \infty$  with probability 1. Let, for some increasing function  $s(x) = o(\sqrt{x/v(x)})$  where a decreasing function  $v(x)$  satisfies  $v(x/2) \leq c_v v(x)$  and  $xv(x) \rightarrow \infty$ ,*

$$\delta v(x) \leq m_1^{[s(x)]}(x) \leq v(x)/\delta, \quad (3.71)$$

for some  $\delta > 0$  and all  $x \geq \hat{x}$ . Assume also

$$\sup_x m_2^{[s(x)]}(x) < \infty, \quad (3.72)$$

and the condition (3.70) hold. Then, for every fixed  $t > 0$  and  $\varepsilon > 0$ , there exists an  $h < \infty$  such that

$$\mathbb{P}_x\left\{-\frac{h}{v(x)} \leq X_n - x \leq \frac{h}{v(x)} \text{ for all } n \leq \frac{t}{v^2(x)}\right\} \geq 1 - \varepsilon$$

for all sufficiently large  $x$ .

*Proof.* Note that (3.71), (3.72) and convergence  $xv(x) \rightarrow \infty$  imply that  $X_n$  satisfies the condition (3.6) with  $r(x) := 2/x$ , for sufficiently large  $\hat{x}$ .

The relation  $s(x) = o(\sqrt{x/v(x)})$  implies  $s(x) = o(1/r(x))$  because  $r(x) = 2/x$  and  $\sqrt{x/v(x)} = o(x)$ , due to  $xv(x) \rightarrow \infty$ . Then Lemma 3.6 is applicable and it suffices to prove the same result for the chain  $Z_n$ , that is, it is sufficient to prove that, for sufficiently large  $h > 0$ ,

$$\mathbb{P}_x \left\{ -\frac{h}{v(x)} \leq Z_n - x \leq \frac{h}{v(x)} \text{ for all } n \leq \frac{t}{v^2(x)} \right\} \geq 1 - \varepsilon \quad (3.73)$$

ultimately in  $x$ .

Since the chain  $Z_n$  satisfies all the conditions of Lemma 3.1,

$$\mathbb{P}_x \left\{ \min_{n \geq 0} Z_n \leq x/2 \right\} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (3.74)$$

Let us center  $Z_n$ , that is, let us consider the process

$$\tilde{Z}_n := Z_n - x - \sum_{j=0}^{n-1} m_1^{[s(Z_j)]}(Z_j), \quad (3.75)$$

which constitutes a martingale with respect to  $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$ . Since

$$\delta \leq \frac{m_1^{[s(z)]}(z)}{v(z)} \leq \frac{1}{\delta}$$

by the condition (3.71), we have

$$\begin{aligned} 0 < \sum_{n=0}^{N-1} m_1^{[s(Z_j)]}(Z_n) &\leq N \frac{1}{\delta} \max_{z > x/2} v(z) \\ &\leq N \frac{c_v}{\delta} v(x) \end{aligned}$$

on the event  $\min_n Z_n > x/2$ , where the last inequality follows from  $v(z) \leq c_v v(x)$ . Hence, for any  $y > 0$ ,

$$\mathbb{P}_x \left\{ \min_n Z_n > x/2, \max_{n \leq N} |Z_n - x| > y \right\} \leq \mathbb{P}_x \left\{ \max_{n \leq N} |\tilde{Z}_n| > y - \frac{c_v}{\delta} N v(x) \right\}.$$

By Kolmogorov's inequality for martingales,

$$\mathbb{P}_x \left\{ \max_{n \leq N} |\tilde{Z}_n| > y - \frac{c_v}{\delta} N v(x) \right\} \leq \frac{\mathbb{E}_x \tilde{Z}_N^2}{(y - c_v N v(x)/\delta)^2}.$$

The second moments of jumps of the martingale  $\tilde{Z}_n$  are bounded by some  $c < \infty$ —see the condition (3.72); therefore,

$$\mathbb{P}_x \left\{ \max_{n \leq N} |\tilde{Z}_n| > y - \frac{c_v}{\delta} N v(x) \right\} \leq \frac{Nc}{(y - c_v N v(x)/\delta)^2}.$$

Taking now  $N = \frac{t}{v^2(x)}$  and  $y = \frac{h}{v(x)}$ , we obtain, for  $h$  sufficiently large,

$$\mathbb{P}_x \left\{ \max_{n \leq N} |\tilde{Z}_n| > y - \frac{c_v}{\delta} N v(x) \right\} \leq \frac{tc}{(h - c_v t / \delta)^2} \leq \frac{\varepsilon}{2},$$

for all sufficiently large  $h$ . Therefore, for those  $n$ ,

$$\mathbb{P}_x \left\{ \min_n Z_n > x/2, \max_{n \leq t/v^2(x)} |Z_n - x| > \frac{h}{v(x)} \right\} \leq \frac{\varepsilon}{2}.$$

Together with (3.74) this completes the proof of (3.73).  $\square$

*Proof of Theorem 3.14.* As shown in Lemma 3.15, it suffices to prove the same result for the chain  $Z_n$ , that is, it is sufficient to prove that

$$\frac{Z_n - x - nv(x)}{\sqrt{nb}} \Rightarrow N_{0,1}. \quad (3.76)$$

as  $x, n \rightarrow \infty$  in such a way that  $n \leq t/v^2(x)$ .

Since the chain  $Z_n$  satisfies all the conditions of Lemma 3.15, for any function  $h(x) \rightarrow \infty$ , given  $Z_0 = x$ ,

$$\mathbb{P}_x \left\{ -\frac{h(x)}{v(x)} \leq Z_n - x \leq \frac{h(x)}{v(x)} \text{ for all } n \leq \frac{t}{v^2(x)} \right\} \rightarrow 1 \text{ as } x \rightarrow \infty. \quad (3.77)$$

The process  $\tilde{Z}_n$  defined in (3.75) constitutes a martingale whose second moments of jumps converge to  $b$  as  $x \rightarrow \infty$ . These observations make it possible to apply the central limit theorem for martingales—see [11] and [28]—and conclude that, given  $Z_0 = x$ , the random sequence

$$\frac{\tilde{Z}_n}{\sqrt{nb}} = \frac{Z_n - x - \sum_{j=0}^{n-1} m_1^{[s(Z_j)]}(Z_j)}{\sqrt{nb}}$$

weakly converges as  $x, n \rightarrow \infty$  to the standard normal distribution.

If  $h(x)/v(x) = o(x)$  then it follows from the condition (3.53) that in the spatial range  $x - h(x)/v(x) \leq y \leq x + h(x)/v(x)$  we have

$$m_1^{[s(y)]}(y) \sim m_1^{[s(x)]}(x) \sim v(x) \text{ as } x \rightarrow \infty.$$

Then within the temporal range  $n \leq t/v^2(x)$ , we deduce from (3.77) that

$$\frac{\sum_{j=0}^{n-1} m_1^{[s(Z_j)]}(Z_j)}{\sqrt{nb}} - \sqrt{n/b} v(x) \xrightarrow{p} 0 \text{ as } x \rightarrow \infty.$$

Therefore,

$$\frac{Z_n - x}{\sqrt{nb}} - \sqrt{n/b} v(x) = \frac{Z_n - x - nv(x)}{\sqrt{nb}}$$

weakly converges as  $x \rightarrow \infty$  to the standard normal distribution and the proof is complete.  $\square$



### 3.10 Global central limit theorem

Let  $v(x)$  be regularly varying decreasing function with index  $-\beta \in [-1, 0]$ , such that

$$v'(x) = O(v(x)/x) \quad \text{as } x \rightarrow \infty. \quad (3.78)$$

We assume that  $X_n$  satisfies the strong law of large numbers,

$$\frac{X_n}{V^{-1}(n)} \xrightarrow{a.s.} 1 \quad \text{as } n \rightarrow \infty; \quad (3.79)$$

for sufficient conditions see Theorem 3.11.

**Theorem 3.16.** *Let, for some increasing function  $s(x) = o(\sqrt{x/v(x)})$ ,*

$$m_1^{[s(x)]}(x) = v(x) + o(\sqrt{v(x)/x}) \quad \text{and} \quad m_2^{[s(x)]}(x) \rightarrow b > 0 \quad \text{as } x \rightarrow \infty \quad (3.80)$$

and

$$\mathbb{P}\{|\xi(x)| > s(x)\} \leq p(x)v(x), \quad (3.81)$$

where  $p(x)$  is an integrable decreasing function. Then

$$\frac{X_n - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}} \Rightarrow N_{0,1} \quad \text{as } n \rightarrow \infty.$$

Some kind of central limit theorems for similar processes were proven by Keller et al. [37, Theorem 2] and Menshikov and Wade [55].

*Proof.* As shown at the beginning of the proof of Lemma 3.15, it suffices to prove the same result for the chain  $Z_n$ , that is, it is sufficient to prove that

$$\frac{Z_n - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}} \Rightarrow N_{0,1} \quad \text{as } n \rightarrow \infty. \quad (3.82)$$

Let us first compute the mean drift of  $V(Z_n)$ : for all sufficiently large  $x$ ,

$$\begin{aligned} m_1^V(x) &:= \mathbb{E}\{V(Z_{n+1}) - V(Z_n) \mid Z_n = x\} \\ &= \mathbb{E}\{V(x + \xi(x)) - V(x); |\xi(x)| \leq s(x)\} \\ &= V'(x)m_1^{[s(x)]}(x) + \frac{1}{2}\mathbb{E}\{V''(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq s(x)\} \\ &= \frac{1}{v(x)}m_1^{[s(x)]}(x) - \frac{1}{2}\mathbb{E}\left\{\frac{v'}{v^2}(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq s(x)\right\}. \end{aligned}$$

Then, owing the condition (3.78), we deduce

$$\begin{aligned} m_1^V(x) &= 1 + o(1/\sqrt{xv(x)}) + O(v'(x)/v^2(x)) \\ &= 1 + o(1/\sqrt{xv(x)}). \end{aligned} \quad (3.83)$$

Further, since  $V'(x+y) = 1/v(x+y) \sim 1/v(x)$  as  $x \rightarrow \infty$  uniformly on  $|y| \leq s(x)$ ,

$$\begin{aligned} m_2^V(x) &:= \mathbb{E}\{(V(x+\xi(x)) - V(x))^2; |\xi(x)| \leq s(x)\} \\ &= \mathbb{E}\{(V'(x+\theta\xi(x))\xi(x))^2; |\xi(x)| \leq s(x)\} \\ &\sim b/v^2(x) \quad \text{as } x \rightarrow \infty. \end{aligned} \quad (3.84)$$

Let us center  $V(Z_n)$ , that is, let us consider

$$\tilde{Z}_n := V(Z_n) - \sum_{j=0}^{n-1} m_1^V(Z_j),$$

so that  $\tilde{Z}_n$  constitutes a martingale with respect to  $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$ . It follows from the strong law of large numbers (3.79) and from (3.84) that

$$v^2(V^{-1}(n))\mathbb{E}\{(\tilde{Z}_{j+1} - \tilde{Z}_j)^2 \mid \mathcal{F}_j\} \xrightarrow{a.s.} b \quad \text{as } j \rightarrow \infty,$$

which implies the convergence

$$\frac{1}{\sigma_n^2} \sum_{j=0}^{n-1} \mathbb{E}\{(\tilde{Z}_{j+1} - \tilde{Z}_j)^2 \mid \mathcal{F}_j\} \xrightarrow{a.s.} 1 \quad \text{as } n \rightarrow \infty, \quad (3.85)$$

where

$$\sigma_n^2 := b \sum_{j=0}^{n-1} \frac{1}{v^2(V^{-1}(j))} \geq c_1 \frac{n}{v^2(V^{-1}(n))} \quad \text{for some } c_1 > 0. \quad (3.86)$$

Since  $|Z_1 - Z_0| \leq s(x)$  given  $Z_0 = x$ ,

$$\begin{aligned} |V(Z_1) - V(Z_0)| &= V'(x + \theta\xi(x))|\xi(x)|\mathbb{I}\{|\xi(x)| \leq s(x)\} \\ &\leq \frac{s(x)}{v(x+s(x))}. \end{aligned}$$

Hence, given  $Z_0 = x + y$ ,

$$|V(Z_1) - V(Z_0)|^2 \leq \psi(x) = o(x/v^3(x))$$

as  $x \rightarrow \infty$  uniformly for all  $|y| \leq x/2$ . Together with the strong law of large numbers (3.79) this implies that there exists a sequence  $\zeta_n$  of random variables such that  $\zeta_n \xrightarrow{a.s.} 0$  as  $n \rightarrow \infty$  and

$$\begin{aligned} |V(Z_{n+1}) - V(Z_n)|^2 &\leq \zeta_n \frac{V^{-1}(n)}{v^3(V^{-1}(n))} \\ &\leq \zeta_n \frac{2n}{v^2(V^{-1}(n))}, \end{aligned}$$

because  $z/v(z) \leq 2V(z)$  by (3.52). Then, by (3.86),

$$|V(Z_{n+1}) - V(Z_n)|^2 \leq \zeta_n \frac{2}{c_1} \sigma_n^2,$$

which allows to conclude that, for any fixed  $\varepsilon > 0$ ,

$$\frac{1}{\sigma_n^2} \sum_{j=0}^{n-1} \mathbb{E}\{(\tilde{Z}_{j+1} - \tilde{Z}_j)^2; |\tilde{Z}_{j+1} - \tilde{Z}_j| \geq \varepsilon \sigma_n \mid \mathcal{F}_j\} \xrightarrow{a.s.} 0 \quad \text{as } n \rightarrow \infty.$$

So, the martingale  $\tilde{Z}_n$  satisfies the conditions of Theorem 4 from [28] and we conclude that

$$\frac{\tilde{Z}_n}{\sigma_n} = \frac{V(Z_n) - \sum_{j=0}^{n-1} m_1^V(Z_j)}{\sigma_n} \Rightarrow N_{0,1} \quad \text{as } n \rightarrow \infty.$$

Further, as follows from the decomposition (3.83) for the mean drift of  $V(Z_n)$

$$\left| \sum_{j=0}^{n-1} m_1^V(Z_j) - n \right| \leq o(1) \sum_{j=0}^{n-1} \frac{1}{\sqrt{Z_j v(Z_j)}}.$$

Therefore, by the strong law of large numbers (3.79), the sum on the right hand side is at the most

$$c_2 \sum_{j=0}^{n-1} \frac{1}{\sqrt{V^{-1}(j) v(V^{-1}(j))}} + \zeta \leq c_3 \frac{n}{\sqrt{V^{-1}(n) v(V^{-1}(n))}} + \zeta$$

where  $c_2, c_3 < \infty$  and  $\zeta$  is a random variable. Since

$$\frac{V^{-1}(n)}{v(V^{-1}(n))} \geq V(V^{-1}(n)) = n$$

by (3.52),

$$\frac{n}{\sqrt{V^{-1}(n) v(V^{-1}(n))}} \leq \frac{\sqrt{n}}{v(V^{-1}(n))}.$$

Combining altogether including the lower bound (3.86) for  $\sigma_n$ , we get

$$\left| \sum_{j=0}^{n-1} m_1^V(Z_j) - n \right| = o(\sigma_n) + \zeta \quad \text{as } n \rightarrow \infty.$$

Thus,

$$\frac{V(Z_n) - n}{\sigma_n} \Rightarrow N_{0,1} \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\frac{Z_n - V^{-1}(n)}{\tilde{\sigma}_n} \Rightarrow N(0, 1),$$

where

$$\begin{aligned}\tilde{\sigma}_n^2 &:= \sigma_n^2((V^{-1})'(n))^2 \\ &= \sigma_n v^2(V^{-1}(n)) \\ &= b \sum_{j=0}^{n-1} \frac{v^2(V^{-1}(n))}{v^2(V^{-1}(j))}.\end{aligned}$$

The sequence  $v^2(V^{-1}(j))$  is regularly varying with index  $-\frac{2\beta}{1+\beta}$ , so that

$$\sum_{j=0}^{n-1} \frac{v^2(V^{-1}(n))}{v^2(V^{-1}(j))} \sim n^{-\frac{2\beta}{1+\beta}} \sum_{j=0}^{n-1} j^{\frac{2\beta}{1+\beta}} \sim \frac{1+\beta}{1+3\beta} n \quad \text{as } n \rightarrow \infty,$$

and the proof is complete.  $\square$

**Theorem 3.17.** *In conditions of Theorem 3.16,*

$$\frac{\max_{k \leq n} X_k - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}} \Rightarrow N_{0,1} \quad \text{as } n \rightarrow \infty.$$

*Proof.* It is again sufficient to prove the same result for the chain  $Z_n$ , that is, it is sufficient to prove that

$$\frac{\max_{k \leq n} Z_k - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}} \Rightarrow N_{0,1} \quad \text{as } n \rightarrow \infty. \quad (3.87)$$

First,  $\sup_{k \leq n} Z_k \geq Z_n$ . Second, the chain  $Z_n$  satisfies all the conditions of Lemma 3.1, so that

$$\limsup_{x \rightarrow \infty} \mathbb{P} \left\{ Z_n \leq \max_{k \leq n} Z_k - h/v(x) \mid \max_{k \leq n} Z_k = x \right\} \leq e^{-\delta \varepsilon h}.$$

By the SLLN for  $Z_n$ ,

$$\frac{\max_{k \leq n} Z_k}{V^{-1}(n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left\{ Z_n \leq \max_{k \leq n} Z_k - h/v(V^{-1}(n)) \right\} \leq e^{-\delta \varepsilon h}.$$

In other words,

$$\frac{\sup_{k \leq n} Z_k - Z_n}{1/v(V^{-1}(n))}$$

is stochastically bounded. Taking into account that  $1/v(V^{-1}(n))$  is regularly varying with index  $\frac{\beta}{1+\beta} < \frac{1}{2}$  we conclude the result.  $\square$

Recall that  $T(x) = \min\{n : X_n > x\}$ .

**Corollary 3.18.** *In conditions of Theorem 3.16,*

$$\frac{T(x) - V(x)}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}} \Rightarrow N_{0,1} \quad \text{as } x \rightarrow \infty.$$

*Proof.* Since  $\{T(x) \leq n\} = \{\sup_{k \leq n} X_k > x\}$ ,

$$\mathbb{P}\left\{\frac{T(x) - V(x)}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}} \leq u\right\} = \mathbb{P}\left\{\sup_{k \leq n} X_k > x\right\}$$

where

$$n := V(x) + u \sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}.$$

Since  $(V^{-1}(z))' = 1/V'(V^{-1}(z)) = v(V^{-1}(z))$  and  $n \sim V(x)$ ,  $(V^{-1}(n))' \sim v(x)$ . Therefore,

$$V^{-1}(n) = x + u \sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v(x)}} + o(\sqrt{x/v(x)}).$$

Hence,

$$\begin{aligned} \mathbb{P}\left\{\sup_{k \leq n} X_k > x\right\} &= \mathbb{P}\left\{\frac{\sup_{k \leq n} X_k - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}} > \frac{x - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}}\right\} \\ &= \mathbb{P}\left\{\frac{\sup_{k \leq n} X_k - V^{-1}(n)}{\sqrt{b \frac{1+\beta}{1+3\beta} n}} > -u + o(1)\right\}, \end{aligned}$$

and Theorem 3.17 completes the proof.  $\square$

### 3.11 Integro-local renewal theorem for transient chain with Normal limit

Notice that  $1/v(x) = o(\sqrt{x/v(x)})$  provided  $xv(x) \rightarrow \infty$ .

**Theorem 3.19.** *Let  $X_n \rightarrow \infty$  with probability 1. Let, for some increasing function  $s(x) = o(\sqrt{x/v(x)})$  where a decreasing function  $v(x)$  satisfies  $v(x/2) \leq c_v v(x)$  and  $xv(x) \rightarrow \infty$ ,*

$$m_1^{[s(x)]}(x) \sim v(x) \quad \text{and} \quad m_2^{[s(x)]}(x) \rightarrow b > 0 \quad \text{as } x \rightarrow \infty, \quad (3.88)$$

$$\mathbb{P}\{|\xi(x)| > s(x)\} \leq p(x)v(x), \quad (3.89)$$

where a decreasing function  $p(x) > 0$  is integrable. Then, for every fixed  $h > 0$  and  $B > 0$ ,

$$\sum_{n=0}^{[B/v^2(x)]} \mathbb{P}_y \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} \sim \frac{h}{v^2(x)} f(B)$$

as  $x \rightarrow \infty$  uniformly for all  $y \in [x, x + o(1/v(x))]$ , where  $f(B) \uparrow 1$  as  $B \rightarrow \infty$ .

*Proof.* Due to the normal approximation provided by Theorem 3.14 we conclude that, for every fixed  $B$ ,

$$\sum_{n=0}^{[B/v^2(x)]} \mathbb{P}_y \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} = \sum_{n=0}^{[B/v^2(x)]} \left( \Phi \left( \frac{h - nv^2(x)}{\sqrt{nbv^2(x)}} \right) - \Phi \left( -\frac{nv^2(x)}{\sqrt{nbv^2(x)}} \right) + o(1) \right)$$

as  $x \rightarrow \infty$  uniformly for all  $y \in [x, x + o(1/v(x))]$ . Approximating the sum on the right by the integral we obtain that its value is equal to

$$\frac{1}{v^2(x)} \int_0^B \left( \Phi \left( \frac{h-z}{\sqrt{bz}} \right) - \Phi \left( -\frac{z}{\sqrt{bz}} \right) \right) dz + o \left( \frac{1}{v^2(x)} \right) \quad (3.90)$$

as  $x \rightarrow \infty$ . The latter integral equals

$$\int_0^B \left( \Phi \left( \frac{h-z}{\sqrt{bz}} \right) - \Phi \left( -\frac{z}{\sqrt{bz}} \right) \right) dz = \int_0^B \frac{dz}{\sqrt{bz}} \int_0^h \varphi \left( \frac{u-z}{\sqrt{bz}} \right) du.$$

Changing the order of integration we obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^h du \int_0^B \frac{1}{\sqrt{bz}} e^{-(u-z)^2/2bz} dz &= \frac{1}{\sqrt{2\pi}} \int_0^h e^{u/b} du \int_0^B \frac{1}{\sqrt{bz}} e^{-u^2/2bz - z/2b} dz \\ &= \frac{2}{b\sqrt{2\pi}} \int_0^{h/b} e^{u/b} du \int_0^B e^{-u^2/2v^2 - v^2/2b^2} dv. \end{aligned}$$

The limit of the internal integral as  $B \rightarrow \infty$  is known—see, e.g. [29, p. 337, 3.325]—and is nothing else but

$$\int_0^\infty e^{-u^2/2v^2 - v^2/2b^2} dv = \frac{b\sqrt{2\pi}}{2} e^{-u/b}.$$

Combining altogether we deduce that

$$\int_0^\infty \left( \Phi \left( \frac{h-z}{\sqrt{bz}} \right) - \Phi \left( -\frac{z}{\sqrt{bz}} \right) \right) dz = h.$$

Together with (3.90) this implies the result.  $\square$

**Theorem 3.20.** *Let conditions of Theorem 3.19 hold. In addition, let (3.88) hold for some increasing function  $s_*(x) = o(1/v(x))$  and*

$$\mathbb{P}\{\xi(x) < -s_*(x)\} \leq p(x)v(x). \quad (3.91)$$

*Then, for every fixed  $h > 0$ , given any initial distribution of  $X_0$ ,*

$$H \left( x, x + \frac{h}{v(x)} \right] \sim \frac{h}{v^2(x)} \quad \text{as } x \rightarrow \infty.$$

*Proof.* Consider the following function

$$r(x) := \frac{x}{b} \int_x^\infty \frac{v(y)}{y^2} dy.$$

Its derivative

$$r'(x) = \frac{1}{b} \int_x^\infty \frac{v(y)}{y^2} dy - \frac{v(x)}{bx} \geq -\frac{v(x)}{bx}$$

satisfies the condition (3.1) for all sufficiently large  $x$  because  $v(x)/x = o(v^2(x))$ . On the other hand, the function  $r(x)$  is decreasing because, by decrease of  $v(y)$ ,

$$r'(x) \leq \frac{v(x)}{b} \int_x^\infty \frac{1}{y^2} dy - \frac{v(x)}{bx} \leq 0.$$

Finally,

$$\begin{aligned} r(x) &\geq \frac{x}{b} \int_x^{2x} \frac{v(y)}{y^2} dy \\ &\geq \frac{xv(2x)}{b} \int_x^{2x} \frac{1}{y^2} dy = \frac{v(2x)}{2b} \geq \frac{c_v v(x)}{2b} \end{aligned} \quad (3.92)$$

and

$$r(x) \leq \frac{xv(x)}{b} \int_x^\infty \frac{1}{y^2} dy = \frac{v(x)}{b}.$$

Then, for all sufficiently large  $x$ ,

$$\frac{2m_1^{[s_*(x)]}(x)}{m_2^{[s_*(x)]}(x)} \geq \frac{v(x)}{b} \geq r(x),$$

and hence the condition (3.6) holds with  $r(x)$  constructed.

We split the proof of the integro-local asymptotics for  $H$  into two parts, upper and lower bounds. First let us prove the right upper bound. By the Markov property it is sufficient to show that

$$\limsup_{x \rightarrow \infty} v^2(x) H_y \left( x, x + \frac{h}{v(x)} \right] \leq h, \quad (3.93)$$

uniformly for  $y > x$ . For any  $A > h$ , by (3.31) and the Markov property,

$$\begin{aligned} H_y \left( x, x + \frac{h}{v(x)} \right] &\leq \mathbb{E}_y \sum_{n=0}^{T\left(x + \frac{A}{v(x)}\right) - 1} \mathbb{I} \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} \\ &\quad + \left( e^{\delta \left( R\left(x + \frac{h}{v(x)}\right) - R\left(x + \frac{A}{v(x)}\right) \right)} + o(1) \right) \sup_z H_z \left( x, x + \frac{h}{v(x)} \right] \end{aligned} \quad (3.94)$$

as  $x \rightarrow \infty$  uniformly for all  $A > h$ . We have

$$\begin{aligned} e^{\delta \left( R\left(x + \frac{A}{v(x)}\right) - R\left(x + \frac{A}{v(x)}\right) \right)} &= e^{-\delta \int_{x+h/v(x)}^{x+A/v(x)} r(y) dy} \\ &\leq e^{-\delta(A-h)r(x+A/v(x))/v(x)} \leq e^{-\delta_0(A-h)}. \end{aligned}$$

The chain  $X_n$  satisfies all the conditions of Theorem 3.3, hence, applying the upper bound of that theorem to the right side of (3.94) we deduce that, for some  $c < \infty$ ,

$$\begin{aligned} H_y\left(x, x + \frac{h}{v(x)}\right] &\leq \mathbb{E}_y \sum_{n=0}^{T\left(x + \frac{A}{v(x)}\right)-1} \mathbb{I}\left\{X_n \in \left(x, x + \frac{h}{v(x)}\right]\right\} \\ &\quad + (e^{-\delta_0(A-h)} + o(1)) \frac{c}{v^2(x)} \quad (3.95) \end{aligned}$$

as  $x \rightarrow \infty$  uniformly for all  $A > h$ . The mean of the sum on the right side of (3.51) may be estimated as follows: for  $C > 1$ ,

$$\begin{aligned} \mathbb{E}_y \sum_{n=0}^{T\left(x + \frac{A}{v(x)}\right)-1} \mathbb{I}\left\{X_n \in \left(x, x + \frac{h}{v(x)}\right]\right\} \\ \leq \mathbb{E}_y \sum_{n=0}^{[CA/v^2(x)]} \mathbb{I}\left\{X_n \in \left(x, x + \frac{h}{v(x)}\right]\right\} \\ + \mathbb{E}_y \left\{ \sum_{n=0}^{T\left(x + \frac{A}{v(x)}\right)-1} \mathbb{I}\left\{X_n > x - \frac{A}{v(x)}\right\}; T\left(x + \frac{A}{v(x)}\right) > \frac{CA}{v^2(x)} \right\}. \end{aligned}$$

The probability of the event over which the second expectation on the right side is taken may be estimated in the following way:

$$\begin{aligned} \mathbb{P}_y \left\{ T\left(x + \frac{A}{v(x)}\right) > \frac{CA}{v^2(x)} \right\} \\ \leq \mathbb{P}_y \left\{ X_n \leq x - \frac{A}{v(x)} \text{ for some } n \geq 0 \right\} \\ + \mathbb{P}_y \left\{ T\left(x + \frac{A}{v(x)}\right) > \frac{CA}{v^2(x)}, X_n > x - \frac{A}{v(x)} \text{ for all } n \leq T\left(x + \frac{A}{v(x)}\right) - 1 \right\}. \end{aligned}$$

The first probability on the right can be made as small as we please by choosing sufficiently large  $A$  and  $x$ , by Lemma 3.1. The second probability possesses the following upper bound which is of Chebyshev's type:

$$\frac{v^2(x)}{CA} \mathbb{E}_y L\left(x - \frac{A}{v(x)}, T\left(x + \frac{A}{v(x)}\right)\right).$$

By Lemma 3.2, the family of random variables

$$v^2(x) L\left(x - \frac{A}{v(x)}, T\left(x + \frac{A}{v(x)}\right)\right)$$



is uniformly integrable, so, for any fixed  $A$ ,

$$\lim_{x \rightarrow \infty} \sup_{y > x} \mathbb{P}_y \left\{ T \left( x + \frac{A}{v(x)} \right) > \frac{CA}{v^2(x)} \right\} \leq \psi(A, C),$$

where  $\psi(A, C) \rightarrow 0$  as  $C \rightarrow \infty$ , for any fixed  $A$ . Therefore, uniform integrability yields

$$\begin{aligned} \lim_{x \rightarrow \infty} \sup_{y > x} v^2(x) \mathbb{E}_y \left\{ \sum_{n=0}^{T(x + \frac{A}{v(x)}) - 1} \mathbb{I} \left\{ X_n > x - \frac{A}{v(x)} \right\}; T \left( x + \frac{A}{v(x)} \right) > \frac{CA}{v^2(x)} \right\} \\ \leq \psi_1(A) + \psi_2(A, C), \end{aligned}$$

where  $\psi_1(A) \rightarrow 0$  as  $A \rightarrow \infty$  and  $\psi_2(A, C) \rightarrow 0$  as  $C \rightarrow \infty$ , for any fixed  $A$ . Hence, uniformly for all  $y$ ,

$$\begin{aligned} \lim_{x \rightarrow \infty} \sup v^2(x) \mathbb{E}_y \sum_{n=0}^{T(x + \frac{A}{v(x)}) - 1} \mathbb{I} \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} \\ \leq \mathbb{E}_y \sum_{n=0}^{[CA/v^2(x)]} \mathbb{I} \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} + \psi_1(A) + \psi_2(A, C), \end{aligned}$$

which being substituted in (3.95) gives

$$\begin{aligned} \lim_{x \rightarrow \infty} \sup v^2(x) H_y \left( x, x + \frac{h}{v(x)} \right] &\leq \lim_{x \rightarrow \infty} \sup v^2(x) \mathbb{E}_y \sum_{n=0}^{[CA/v^2(x)]} \mathbb{I} \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} \\ &\quad + \psi_2(A, C) + ce^{-\delta_0(A-h)} + \psi_1(A). \end{aligned}$$

As already shown, as  $x \rightarrow \infty$ ,

$$v^2(x) \sum_{n=0}^{[CA/v^2(x)]} \mathbb{P}_y \left\{ X_n \in \left( x, x + \frac{h}{v(x)} \right] \right\} \rightarrow hf(CA),$$

which implies the following upper bound, for each fixed  $A, C > 1$ ,

$$\lim_{x \rightarrow \infty} \sup v^2(x) H_y \left( x, x + \frac{h}{v(x)} \right] \leq hf(CA) + \psi_2(A, C) + ce^{-\delta_0(A-h)} + \psi_1(A).$$

Letting now first  $C \rightarrow \infty$  and then  $A \rightarrow \infty$ , we get the required upper bound (3.93).

Now proceed with lower bound. First notice that, by Theorem 3.19,

$$\liminf_{x \rightarrow \infty} v^2(x) H_y \left( x, x + \frac{h}{v(x)} \right] \geq h \quad (3.96)$$

as  $x \rightarrow \infty$  uniformly for all  $y \in [x, x + o(1/v(x))]$ . It remains to prove that (3.96) holds for any fixed  $y$ . By the Markov property, it suffices to show that there exists

a decreasing function  $\widehat{s}(x) = o(1/v(x))$  such that the overshoot over the level  $x$  is less than  $\widehat{s}(x)$  with high probability, that is,

$$\mathbb{P}_y\{X_{T(x)} - x > \widehat{s}(x)\} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (3.97)$$

Indeed,

$$\begin{aligned} \mathbb{P}_y\{X_{T(x)} - x > \widehat{s}(x)\} &\leq \sum_{n=1}^{\infty} \int_0^x \mathbb{P}_y\{X_n \in dz\} \mathbb{P}\{z + \xi(z) > x + \widehat{s}(x)\} \\ &= \int_0^x \mathbb{P}\{z + \xi(z) > x + \widehat{s}(x)\} H_y(dz) = \int_0^{x/2} + \int_{x/2}^x. \end{aligned} \quad (3.98)$$

By Chebyshev's inequality, the first integral on the right is bounded above by

$$\begin{aligned} \int_0^{x/2} \mathbb{P}\{\xi(z) > x/2 + \widehat{s}(x)\} H_y(dz) &\leq \int_0^{x/2} \frac{c}{(x/2 + \widehat{s}(x))^2} H_y(dz) \\ &\leq \frac{cH[0, x/2]}{(x/2 + \widehat{s}(x))^2} \\ &\leq \frac{c_1}{(x/2)^2 r(x/2) v(x/2)} \\ &\leq \frac{c_2}{(x/2)^2 v^2(x/2)} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \end{aligned}$$

due to Theorem 3.3, the inequality (3.92) and the convergence  $xv(x) \rightarrow \infty$ . Let  $\widehat{s}(z) > s_*(z)$ , then by Chebyshev's inequality,

$$\begin{aligned} \mathbb{P}\{\xi(z) > x + \widehat{s}(x) - z\} &\leq \mathbb{P}\{\xi(z) > s(z)\} + \frac{\mathbb{E}\{\xi^2(z); \xi(z) \in (\widehat{s}(z), s(z))\}}{(x + \widehat{s}(x) - z)^2} \\ &\leq \mathbb{P}\{\xi(z) > s(z)\} + \frac{\mathbb{E}\{\xi^2(z); \xi(z) \in (s_*(z), s(z))\}}{(x + \widehat{s}(x) - z)^2} \\ &\leq p(z)v(z) + \frac{\varepsilon(z)}{(x + \widehat{s}(x) - z)^2}, \end{aligned}$$

where  $\varepsilon(z) \rightarrow 0$  as  $z \rightarrow \infty$ ; the last inequality follows by the condition (3.89) and because

$$\begin{aligned} \varepsilon(z) &:= \mathbb{E}\{\xi^2(z); \xi(z) \in (s_*(z), s(z))\} \\ &= \mathbb{E}\{\xi^2(z); |\xi(z)| \leq s(z)\} - \mathbb{E}\{\xi^2(z); |\xi(z)| \leq s_*(z)\} \\ &\rightarrow b - b = 0 \quad \text{as } z \rightarrow \infty, \end{aligned}$$

by the condition (3.88) which is fulfilled for both  $s(z)$  and  $s_*(z)$ . Therefore, the second integral on the right of (3.98) is not greater than

$$\begin{aligned} &\int_{x/2}^x \left( p(z)v(z) + \frac{\varepsilon(z)}{(x + \widehat{s}(x) - z)^2} \right) H_y(dz) \\ &\leq \int_{x/2}^{\infty} p(z)v(z) H_y(dz) + \sup_{z > x/2} \varepsilon(z) \int_{x/2}^x \frac{H_y(dz)}{(x + \widehat{s}(x) - z)^2 v(z)}. \end{aligned}$$

The first integral on the right goes to zero as  $x \rightarrow \infty$ , see calculations leading to (3.30). Similar calculations also allow to bound the second integral: for some  $c_3 < \infty$ ,

$$\begin{aligned} \int_{x/2}^x \frac{H_y(dz)}{(x + \widehat{s}(x) - z)^2} &\leq c_3 \int_{x/2}^x \frac{dz}{(x + \widehat{s}(x) - z)^2 v(z)} \\ &\leq \frac{c_3}{v(x)} \int_{x/2}^x \frac{dz}{(x + \widehat{s}(x) - z)^2} \\ &\leq \frac{c_3}{v(x) \widehat{s}(x)} \end{aligned}$$

If we choose  $\widehat{s}(x) = o(1/v(x))$  in such a way that

$$\frac{\sup_{z > x/2} \varepsilon(z)}{v(x) \widehat{s}(x)} \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

then altogether yields the bound (3.97) for the overshoot. That concludes the proof.  $\square$

Theorem 3.19 and the proof of Theorem 3.20 imply the following result.

**Theorem 3.21.** *Let conditions of Theorems 3.19 and 3.20 hold. Then, for every fixed  $h > 0$ ,*

$$\sum_{k=0}^n \mathbb{P}_y \left\{ X_k \in \left( x, x + \frac{h}{v(x)} \right] \right\} = \frac{h}{v^2(x)} f(nv^2(x)) + o\left(\frac{1}{v^2(x)}\right)$$

as  $x \rightarrow \infty$  uniformly for all  $y \in [x, x + o(1/v(x))]$  and for all  $n \geq 1$ , where  $f(z) \uparrow 1$  as  $z \rightarrow \infty$ .

**Theorem 3.22.** *Let conditions of Theorems 3.16 and 3.20 hold. Then, given any initial distribution of  $X_0$  and any fixed  $h > 0$ ,*

$$\sum_{k=0}^n \mathbb{P} \left\{ X_k \in \left( x, x + \frac{h}{v(x)} \right] \right\} = \frac{h}{v^2(x)} \Phi \left( \frac{n - V(x)}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}} \right) + o\left(\frac{1}{v^2(x)}\right) \quad (3.99)$$

as  $x \rightarrow \infty$  uniformly for all  $n \geq 1$ .

*Proof.* We have

$$\sum_{k=0}^n \mathbb{P} \left\{ X_k \in \left( x, x + \frac{h}{v(x)} \right] \right\} = \mathbb{E} \sum_{k=T(x)}^n \mathbb{I} \left\{ X_k \in \left( x, x + \frac{h}{v(x)} \right] \right\}.$$

As (3.97) shows,  $v(x)(X_{T(x)} - x) \rightarrow 0$  in probability. This allows to apply Theorem 3.21: as  $x \rightarrow \infty$ ,

$$\mathbb{E} \sum_{k=T(x)}^n \mathbb{I} \left\{ X_k \in \left( x, x + \frac{h}{v(x)} \right] \right\} = \frac{h}{v^2(x)} \mathbb{E} f(v^2(x)(n - T(x))^+) + o\left(\frac{1}{v^2(x)}\right).$$

Further, fix  $u \in \mathbb{R}$  and take

$$n = V(x) + u \sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}.$$

Then

$$\begin{aligned} v^2(x)(n - T(x))^+ &= O(\sqrt{xv(x)}) \frac{(n - T(x))^+}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}} \\ &= O(\sqrt{xv(x)}) \left( u + \frac{V(x) - T(x)}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}} \right)^+. \end{aligned}$$

Since  $xv(x) \rightarrow \infty$ , the latter quantity goes to infinity with probability

$$\mathbb{P} \left\{ \frac{V(x) - T(x)}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{v^3(x)}}} > -u \right\} \approx \Phi(u)$$

and equals zero with probability going to  $1 - \Phi(u)$ , both by Corollary 3.18. Taking into account that  $f(z) \rightarrow 1$  as  $z \rightarrow \infty$ , we conclude that

$$\mathbb{E} f(v^2(x)(n - T(x))^+) \rightarrow \Phi(u) \quad \text{as } x \rightarrow \infty,$$

which completes the proof. □

## Chapter 4

# Doob's $h$ -transform: transition from recurrent to transient chains and vice versa

### 4.1 Doob's $h$ -transform for transition kernels

#### 4.1.1 General change of measure methodology for transition kernels

Let  $P(x, A) : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}^+$  be a positive transition kernel in  $\mathbb{R}$  which is not necessarily stochastic. Let  $U(x) > 0$  be a positive function such that

$$\int_{\mathbb{R}} U(y) P(x, dy) < \infty \quad \text{for all } x \in \mathbb{R}.$$

Then it allows to define a new transition kernel

$$Q(x, A) := \int_A \frac{U(y)}{U(x)} P(x, dy),$$

which is exactly Doob's  $h$ -transform for  $P$  with weight function  $U$ . If  $P$  is a transition probability and  $U$  is a harmonic function for  $P$ , then  $Q$  is a transition probability too.

Standard calculations

$$\begin{aligned} Q^n(x, A) &= \int_{\mathbb{R}} Q(x, dy_1) \dots \int_{\mathbb{R}} Q(y_{n-2}, dy_{n-1}) \int_A Q(y_{n-1}, dy_n) \\ &= \int_{\mathbb{R}} \frac{U(y_1)}{U(x)} P(x, dy_1) \dots \int_{\mathbb{R}} \frac{U(y_{n-1})}{U(y_{n-2})} P(y_{n-2}, dy_{n-1}) \int_A \frac{U(y_n)}{U(y_{n-1})} P(y_{n-1}, dy_n) \\ &= \int_{\mathbb{R}} \frac{U(y_n)}{U(x)} P(x, dy_1) \dots \int_{\mathbb{R}} P(y_{n-2}, dy_{n-1}) \int_A P(y_{n-1}, dy_n) \\ &= \int_A \frac{U(y_n)}{U(x)} P^n(x, dy_n) \end{aligned}$$

show that Doob's  $h$ -transform of the  $n$ th power of  $P$ ,  $P^n$ , is equal to the  $n$ th power of Doob's  $h$ -transform of  $P$ ,  $Q^n$ . Similarly, for any collection of Borel sets  $A_1, \dots, A_n$ ,

$$\begin{aligned} & \int_{A_1} Q(x, dy_1) \dots \int_{A_{n-1}} Q(y_{n-2}, dy_{n-1}) \int_{A_n} Q(y_{n-1}, dy_n) \\ &= \int_{A_1} P(x, dy_1) \dots \int_{A_{n-1}} P(y_{n-2}, dy_{n-1}) \int_{A_n} \frac{U(y_n)}{U(x)} P(y_{n-1}, dy_n). \end{aligned}$$

Performing the inverse change of measure we get

$$P^n(x, dy) = \frac{U(x)}{U(y)} Q^n(x, dy) \quad (4.1)$$

and

$$\begin{aligned} & \int_{A_1} P(x, dy_1) \dots \int_{A_{n-1}} P(y_{n-2}, dy_{n-1}) \int_{A_n} P(y_{n-1}, dy_n) \\ &= \int_{A_1} Q(x, dy_1) \dots \int_{A_{n-1}} Q(y_{n-2}, dy_{n-1}) \int_{A_n} \frac{U(x)}{U(y_n)} Q(y_{n-1}, dy_n). \end{aligned} \quad (4.2)$$

Denote

$$q(x) := -\log Q(x, \mathbb{R}^+).$$

Let us consider the following normalised kernel

$$\hat{P}(x, dy) = \frac{Q(x, dy)}{Q(x, \mathbb{R}^+)} = Q(x, dy) e^{q(x)}$$

and let  $\{\hat{X}_n\}$  be a Markov chain with this transition probability. Then

$$Q(x, dy) = \hat{P}(x, dy) e^{-q(x)}$$

and hence, by (4.1), we arrive at the following basic equalities:

$$P^n(x, dy) = \frac{U(x)}{U(y)} \mathbb{E}_x \{ e^{-\sum_{k=0}^{n-1} q(\hat{X}_k)}; \hat{X}_n \in dy \} \quad (4.3)$$

and

$$\begin{aligned} & \int_{A_1} P(x, y_1) \dots \int_{A_{n-1}} P(y_{n-2}, dy_{n-1}) P(y_{n-1}, dy_n) \\ &= \frac{U(x)}{U(y_n)} \mathbb{E}_x \{ e^{-\sum_{k=0}^{n-1} q(\hat{X}_k)}; \hat{X}_1 \in A_1, \dots, \hat{X}_{n-1} \in A_{n-1}, \hat{X}_n \in dy \}. \end{aligned} \quad (4.4)$$

### 4.1.2 Application to killed Markov chain

In this subsection we specify the above techniques for the case which we are most interested in—transition kernel corresponding to a Markov chain killed at entering some fixed set. Namely, let  $X_n$  be a Markov chain with transition probability  $P(\cdot, \cdot)$ ,  $B \subset \mathbb{R}$  be some fixed set and  $\tau_B := \min\{n \geq 1 : X_n \in B\}$ . Consider a substochastic transition kernel

$$P_B(x, A) := P(x, A \setminus B) = \mathbb{P}_x\{X_1 \in A, \tau_B > 1\},$$

which is the transition kernel corresponding to  $X_n$  killed at entering  $B$ .

Given a weight function  $U(x) > 0$ , the corresponding change of measure produces a transition kernel  $Q$  which may be rewritten as follows

$$\begin{aligned} Q(x, dy) &:= \frac{U(y)}{U(x)} \mathbb{P}_x\{X_1 \in dy, \tau_B > 1\} \\ &= \frac{U(y)}{U(x)} \mathbb{P}_x\{X_1 \in dy, X_1 \notin B\}. \end{aligned} \quad (4.5)$$

Consequently, performing the inverse change of measure we arrive at the following basic equality:

$$\begin{aligned} \mathbb{P}_x\{X_n \in dy, \tau_B > n\} &= \frac{U(x)}{U(y)} Q^n(x, dy) \\ &= \frac{U(x)}{U(y)} \mathbb{E}_x\{e^{-\sum_{k=0}^{n-1} q(\hat{X}_k)}; \hat{X}_n \in dy\}, \end{aligned} \quad (4.6)$$

where

$$q(x) := -\log \int_{\mathbb{R} \setminus B} \frac{U(y)}{U(x)} P(x, dy) \quad (4.7)$$

and  $\{\hat{X}_n\}$  is a Markov chain with transition probability

$$\hat{P}(x, A) = \frac{Q(x, A)}{Q(x, \mathbb{R}^+)} = \frac{\int_{A \setminus B} U(y) P(x, dy)}{\int_{\mathbb{R} \setminus B} U(y) P(x, dy)}. \quad (4.8)$$

## 4.2 Increasing drift via change of measure with weight function close to harmonic function

### 4.2.1 Stochastic kernel

Let  $X_n$  be a Markov chain with jumps  $\xi(x)$ . Let, for some increasing function  $s(x)$  and decreasing function  $r(x) \rightarrow 0$  as  $x \rightarrow \infty$ ,

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \sim -c_{12}r(x), \quad c_{12} \in \mathbb{R}, \quad (4.9)$$

$$m_2^{[s(x)]}(x) \rightarrow b > 0. \quad (4.10)$$

If we need to increase the drift—say if we need to pass from a recurrent Markov chain to a transient one, then clearly an increasing weight should be applied. So, let  $U(x) \geq 0$  be an increasing differentiable function such that, for some  $c_U > 0$ ,

$$\frac{U'(x)}{U(x)} \sim c_U r(x) \quad \text{as } x \rightarrow \infty \quad (4.11)$$

and

$$U(x+y) \sim U(x) \quad \text{and} \quad U'(x+y) \sim U'(x) \quad (4.12)$$

as  $x \rightarrow \infty$  uniformly for all  $|y| \leq s(x)$ .

We assume that  $U$  is close to a harmonic function in the following sense:

$$\mathbb{E}_x U(X_1) = \mathbb{E}U(x + \xi(x)) \sim U(x) \quad \text{as } x \rightarrow \infty. \quad (4.13)$$

This condition provides asymptotic stochasticity of  $Q$ , that is,  $Q(x, \mathbb{R}) \rightarrow 1$  as  $x \rightarrow \infty$ .

Let  $Q$ ,  $\widehat{P}(\cdot, \cdot)$ ,  $\widehat{X}_n$ , and  $\widehat{\xi}(x)$  be defined for  $P(\cdot, \cdot)$  with weight function  $U$  as described in the last section.

**Lemma 4.1.** *Let conditions (4.9)–(4.13) hold. Then*

$$\mathbb{E}\{\widehat{\xi}(x); |\widehat{\xi}(x)| \leq s(x)\} \sim (c_U - c_{12}/2)br(x), \quad (4.14)$$

$$\mathbb{E}\{(\widehat{\xi}(x))^2; |\widehat{\xi}(x)| \leq s(x)\} \rightarrow b \quad (4.15)$$

as  $x \rightarrow \infty$ , so hence

$$\frac{2\widehat{m}_1^{[s(x)]}(x)}{\widehat{m}_2^{[s(x)]}(x)} \sim (2c_U - c_{12})r(x).$$

In addition,

$$\mathbb{P}\{\widehat{\xi}(x) < -s(x)\} \leq (1 + o(1))\mathbb{P}\{\xi(x) < -s(x)\}, \quad (4.16)$$

$$\mathbb{P}\{\widehat{\xi}(x) > s(x)\} \leq (1 + o(1)) \frac{\mathbb{E}\{U(x + \xi(x)); \xi(x) > s(x)\}}{U(x)}. \quad (4.17)$$

*Proof.* By the construction of  $\widehat{X}_n$  and the condition (4.13),

$$\begin{aligned} \mathbb{E}\{\widehat{\xi}(x); |\widehat{\xi}(x)| \leq s(x)\} &= \frac{\mathbb{E}\{U(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\}}{\mathbb{E}U(x + \xi(x))} \\ &\sim \frac{\mathbb{E}\{U(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\}}{U(x)}. \end{aligned}$$

By Taylor's theorem,

$$\begin{aligned} \mathbb{E}\{U(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\} &= U(x)\mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{U'(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq s(x)\}, \end{aligned}$$



where  $\theta = \theta(x, \xi(x)) \in (0, 1)$ . The first term on the right side is equivalent to  $-c_{12}bU(x)r(x)/2$ , as follows from (4.9) and (4.10). By the condition (4.12),  $U'(x + \theta\xi(x)) \sim U'(x)$  as  $x \rightarrow \infty$  uniformly for all  $|\xi(x)| \leq s(x)$  which implies, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{E}\{U'(x + \theta\xi(x))\xi^2(x); |\xi(x)| \leq s(x)\} &\sim U'(x)\mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\} \\ &\sim U'(x)b \sim c_U br(x)U(x), \end{aligned}$$

due to the conditions (4.10) and (4.11). Altogether yields that

$$\mathbb{E}\{U(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\} \sim (c_u - c_{12}/2)br(x)U(x) \quad \text{as } x \rightarrow \infty,$$

and (4.14) follows. The second result, (4.15), follows by similar arguments starting with the following

$$\begin{aligned} \mathbb{E}\{\widehat{\xi}^2(x); |\widehat{\xi}(x)| \leq s(x)\} &= \frac{\mathbb{E}\{U(x + \xi(x))\xi^2(x); |\xi(x)| \leq s(x)\}}{\mathbb{E}U(x + \xi(x))} \\ &\sim \frac{\mathbb{E}\{U(x + \xi(x))\xi^2(x); |\xi(x)| \leq s(x)\}}{U(x)}. \end{aligned}$$

Using (4.13) and recalling that  $U$  is increasing, we also get

$$\begin{aligned} \mathbb{P}\{\widehat{\xi}(x) < -s(x)\} &\sim \frac{\mathbb{E}\{U(x + \xi(x)); \xi(x) < -s(x)\}}{U(x)} \\ &\leq \mathbb{P}\{\xi(x) < -s(x)\}. \end{aligned}$$

The last assertion, (4.17), follows again by (4.13), and the proof of the lemma is complete.  $\square$

### 4.2.2 Killed Markov chain

Let  $\widehat{x} \in \mathbb{R}^+$  be some level. For  $X_n$  killed at entering  $B := (-\infty, \widehat{x}]$ , let us perform change of measure with increasing weight function  $U$  and consider the corresponding kernel  $Q$ ,

$$Q(x, A) = \frac{\mathbb{E}\{U(x + \xi(x)); x + \xi(x) \in A \cap (\widehat{x}, \infty)\}}{U(x)}, \quad (4.18)$$

and embedded Markov chain  $\widehat{X}_n$  with transition probabilities

$$\widehat{P}(x, A) = \frac{\mathbb{E}\{U(x + \xi(x)); x + \xi(x) \in A \cap (\widehat{x}, \infty)\}}{\mathbb{E}\{U(x + \xi(x)); x + \xi(x) > \widehat{x}\}}, \quad (4.19)$$

if  $\mathbb{P}\{x + \xi(x) > \widehat{x}\} > 0$  and  $\widehat{P}(x, A) = \mathbb{I}\{x \in A\}$  otherwise. Let  $\widehat{\xi}(x)$  be jumps of  $\widehat{X}_n$ .

The following result is almost immediate from Lemma 4.1.

**Lemma 4.2.** *Let conditions (4.9)–(4.13) hold and, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\xi(x); x + \xi(x) \leq \hat{x}\} = o(r(x)), \quad (4.20)$$

$$\mathbb{E}\{\xi^2(x); x + \xi(x) \leq \hat{x}\} \rightarrow 0. \quad (4.21)$$

*Then conclusions (4.14)–(4.17) hold.*

*If, in addition, (4.20) and (4.21) hold for any  $\hat{x}$  and  $c_U > c_{12}/2$ , then there exists a sufficiently large  $\hat{x}$  such that the jumps of the corresponding Markov chain  $\hat{X}_n$  satisfies the inequality*

$$\mathbb{E}\{\hat{\xi}(x); |\hat{\xi}(x)| \leq s(x)\} \geq \frac{c_U - c_{12}/2}{2} br(x) \quad \text{for all } x \geq \hat{x}. \quad (4.22)$$

*Proof.* By the condition (4.20),

$$\mathbb{E}\{U(x + \xi(x)); x + \xi(x) \leq \hat{x}\} \leq U(\hat{x})\mathbb{P}\{x + \xi(x) \leq \hat{x}\} = o(r(x))$$

as  $x \rightarrow \infty$ , so hence

$$\mathbb{E}\{U(x + \xi(x)); x + \xi(x) > \hat{x}\} \sim \mathbb{E}U(x + \xi(x)) \sim U(x),$$

owing to (4.13). By the same arguments,

$$\begin{aligned} \mathbb{E}\{U(x + \xi(x))\xi^2(x); x + \xi(x) > \hat{x}, |\xi(x)| \leq s(x)\} \\ = \mathbb{E}\{U(x + \xi(x))\xi^2(x); |\xi(x)| \leq s(x)\} + o(1), \end{aligned}$$

$$\begin{aligned} \mathbb{E}\{U(x + \xi(x))\xi(x); x + \xi(x) > \hat{x}, |\xi(x)| \leq s(x)\} \\ = \mathbb{E}\{U(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\} + o(r(x)), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\{U(x + \xi(x))\xi(x); x + \xi(x) > \hat{x}, |\xi(x)| \leq s(x)\} \\ \geq \mathbb{E}\{U(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\}, \end{aligned}$$

which completes the proof because of Lemma 4.1.  $\square$

## 4.3 Decreasing drift via change of measure with weight function close to harmonic function

### 4.3.1 Stochastic kernel

In this section let  $X_n$  be a Markov chain such that, for some increasing function  $s(x)$  and decreasing function  $r(x) \rightarrow 0$  as  $x \rightarrow \infty$ ,

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \sim r(x), \quad (4.23)$$

$$m_2^{[s(x)]}(x) \rightarrow b > 0. \quad (4.24)$$

If we need to decrease the drift—say if we need to pass from a transient Markov chain to a recurrent one, then clearly a decreasing weight should be applied. So, let  $U(x) \geq 0$  be a decreasing differentiable function such that, for some  $c_U < 0$ , (4.11) and (4.12) hold. As in the previous section, we again assume that  $U$  is close to a harmonic function in the sense (4.13).

In the same way as Lemma (4.1), the following result follows.

**Lemma 4.3.** *Let conditions (4.23), (4.24) and (4.11)–(4.13) hold. Then*

$$\mathbb{E}\{\widehat{\xi}(x); |\widehat{\xi}(x)| \leq s(x)\} \sim (c_U + 1/2)br(x), \quad (4.25)$$

$$\mathbb{E}\{(\widehat{\xi}(x))^2; |\widehat{\xi}(x)| \leq s(x)\} \rightarrow b \quad (4.26)$$

as  $x \rightarrow \infty$ , so hence

$$\frac{2\widehat{m}_1^{[s(x)]}(x)}{\widehat{m}_2^{[s(x)]}(x)} \sim (2c_U + 1)r(x). \quad (4.27)$$

In addition,

$$\mathbb{P}\{\widehat{\xi}(x) > s(x)\} \leq (1 + o(1))\mathbb{P}\{\xi(x) > s(x)\}, \quad (4.28)$$

$$\mathbb{P}\{\widehat{\xi}(x) < -s(x)\} \leq (1 + o(1)) \frac{\mathbb{E}\{U(x + \xi(x)); \xi(x) < -s(x)\}}{U(x)}. \quad (4.29)$$

### 4.3.2 Killed Markov chain

Let  $\widehat{x} \in \mathbb{R}^+$  be some level. For  $X_n$  killed at entering  $B := (-\infty, \widehat{x}]$ , let us perform change of measure with decreasing weight function  $U$  and consider the corresponding kernel  $Q$  and embedded Markov chain  $\widehat{X}_n$

Then similarly to Lemma 4.2 we get the following result.

**Lemma 4.4.** *Let the conditions (4.23), (4.24), and (4.11)–(4.13) hold. Then conclusions (4.25)–(4.29) hold.*

## 4.4 Cycle structure of Markov chain and Doob's transform

Let a Markov chain  $X_n$  possesses a sigma-finite invariant measure  $\pi$  which corresponds to either positive (finite  $\pi$ ) or null (sigma-finite  $\pi$ ) recurrence. The following representation for the invariant measure  $\pi$  via cycle structure of the Markov chain  $X_n$  is well known—see, e.g. [56, Theorem 10.4.9]—with  $B = (-\infty, \widehat{x}]$ ,

$$\pi(dy) = \int_B \pi(dz) \sum_{n=0}^{\infty} \mathbb{P}_z\{X_n \in dy; \tau_B > n\}, \quad y \geq 0. \quad (4.30)$$

Substituting here (4.6), we get

$$\begin{aligned}\pi(dy) &= \frac{1}{U(y)} \int_B \pi(dz) U(z) \sum_{n=0}^{\infty} \mathbb{E}_z \{ e^{-\sum_{k=0}^{n-1} q(\hat{X}_k)}; \hat{X}_n \in dy \} \\ &= \frac{\hat{H}^{(q)}(dy)}{U(y)} \int_B \pi(dz) U(z),\end{aligned}$$

where the weighted renewal measure  $H^{(q)}$  for  $\hat{X}_n$  is defined as

$$\hat{H}^{(q)}(dx) = \sum_{n=0}^{\infty} \mathbb{E} \{ e^{-\sum_{k=0}^{n-1} q(\hat{X}_k)}; \hat{X}_n \in dx \}, \quad (4.31)$$

with chain  $\hat{X}_n$  having initial distribution

$$\mathbb{P}\{\hat{X}_0 \in dz\} = \frac{\pi(dz)U(z)}{c^*}, \quad z \in B, \quad \text{and} \quad c^* := \int_B U(z)\pi(dz). \quad (4.32)$$

Therefore, in particular,

$$\pi(x_1, x_2] = c^* \int_{x_1}^{x_2} \frac{H^{(q)}(dy)}{U(y)}. \quad (4.33)$$

So, the main idea for investigation of the invariant measure is to identify an increasing test function  $U(x)$  which is sufficiently close to a harmonic function in a sense that its drift is sufficiently small for large  $x$  which implies small values of  $q(x)$ . We also need to choose  $U(x)$  in such a way that the chain  $\hat{X}_n$  is transient. Then factorisation result for the renewal function  $H^{(q)}$  with weights, see Section 3.3, and an integro-local renewal theorem for  $\hat{X}_n$  allow to derive asymptotics for the tail of  $\pi$ .

## Chapter 5

# Tail analysis for recurrent Markov chains with drift proportional to $1/x$

### 5.1 Stationary measure of recurrent chains: power-like asymptotics

In this chapter we assume that the Markov chain  $X_n$  is recurrent and possesses a stationary (invariant) measure which is either probabilistic in the case of positive recurrence or  $\sigma$ -finite in the case of null recurrence. We denote this measure by  $\pi$ .

If we consider an irreducible aperiodic Markov chain on  $\mathbb{Z}^+$ , then existence of probabilistic invariant measure is equivalent to finiteness of  $\mathbb{E}_0\tau_0$  where  $\tau_0 := \min\{n \geq 1 : X_n = 0\}$ . The case of null recurrence corresponds to almost finite  $\tau_0$  with infinite mean,  $\mathbb{E}\tau_0 = \infty$ . For the state space  $\mathbb{R}^+$ , the standard conditions for recurrence are Harris recurrence and strongly aperiodicity of  $X_n$ , see related definitions in [56].

We consider the case where  $\pi$  has unbounded support, that is,  $\pi(x, \infty) > 0$  for every  $x$ . Our main problem is to describe the asymptotic behaviour of its tail,  $\pi(x, \infty)$ , for a class of Markov chains with asymptotically zero drift.

As was shown in [46, Theorem 1] any positive recurrent Markov chain with asymptotically zero drift has heavy-tailed invariant distribution provided

$$\liminf_{x \rightarrow \infty} \mathbb{E}\{\xi^2(x); \xi(x) > 0\} > 0;$$

that is, all positive exponential moments of the invariant distribution are infinite. This section is devoted to the precise asymptotic behaviour of the invariant measure in the critical case where the drift behaves like  $c/x$  for large  $x$ .

There are two types of Markov chains for which the invariant measure is explicitly calculable. Both are related to skip-free processes, either on lattice or on continuous state space  $\mathbb{R}^+$ .

The first case where the stationary distribution is explicitly known is diffusion processes on  $\mathbb{R}^+$  (slotted in time if we need just a Markov chain), see (2.1).

The second case is the Markov chain on  $\mathbb{Z}^+$  with  $\xi(x)$  taking values  $-1$ ,  $1$  and  $0$  only, with probabilities  $p_-(x)$ ,  $p_+(x)$  and  $1 - p_-(x) - p_+(x)$  respectively,  $p_-(0) = 0$ . Then the stationary measure  $\pi(x)$ ,  $x \in \mathbb{Z}^+$ , satisfies the equations

$$\pi(x) = \pi(x-1)p_+(x-1) + \pi(x)(1 - p_+(x) - p_-(x)) + \pi(x+1)p_-(x+1),$$

which have the following solution:

$$\pi(x) = \pi(0) \prod_{k=1}^x \frac{p_+(k-1)}{p_-(k)} = \pi(0) e^{\sum_{k=1}^x (\log p_+(k-1) - \log p_-(k))}, \quad (5.1)$$

where under some regularity conditions the sum may be approximated by the integral like in the diffusion case.

To the best of our knowledge there are no other results in the literature on the exact asymptotic behaviour for the measure  $\pi$ .

In the whole chapter we consider a recurrent Markov chain  $X_n$  whose jumps are such that

$$m_2^{[s(x)]}(x) \rightarrow b > 0 \quad \text{and} \quad m_1^{[s(x)]}(x)x \rightarrow -\mu \in \mathbb{R} \quad \text{as } x \rightarrow \infty, \quad (5.2)$$

where a function  $s(x) = o(x)$  is increasing and  $\mu > -b/2$ ; the case  $\mu \in (-b/2, b/2)$  usually corresponds to null recurrence of  $X_n$  while  $\mu > b/2$  corresponds to positive recurrence; in the case  $\mu = b/2$  either null or positive recurrence can happen, see [51, Sections 2.2, 2.4], or Corollaries 2.3, 2.14. In addition, we assume that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = -r(x) + o(p(x)) \quad \text{as } x \rightarrow \infty \quad (5.3)$$

for some monotone function  $r(x) \rightarrow 0$  satisfying  $r(x)x \rightarrow 2\mu/b > -1$  as  $x \rightarrow \infty$  and some decreasing integrable function  $p(x) \in [0, r(x)]$ . Since  $p(x)$  is decreasing and integrable,  $p(x)x \rightarrow 0$  as  $x \rightarrow \infty$ . We also assume that

$$r'(x) = O(1/x^2) \quad \text{and} \quad p'(x) = O(1/x^2). \quad (5.4)$$

An equivalent way to state the condition (5.3) is

$$m_1^{[s(x)]}(x) + \frac{m_2^{[s(x)]}(x)}{2} r(x) = o(p(x)) \quad \text{as } x \rightarrow \infty. \quad (5.5)$$

Define a monotone function

$$R(x) := \int_0^x r(y) dy. \quad (5.6)$$

Since  $xr(x) \rightarrow 2\mu/b > -1$ ,

$$\frac{R(x)}{\log x} \rightarrow \frac{2\mu}{b} > -1 \quad \text{as } x \rightarrow \infty.$$

Define the following increasing function which plays the most important role in our analysis of recurrent Markov chain:

$$U(x) := \int_0^x e^{R(y)} dy \rightarrow \infty \quad \text{as } x \rightarrow \infty, \quad (5.7)$$

again due to  $2\mu/b > -1$ ; in what follows we show that the function  $U(x)$  is very close to be a harmonic function. Note that the function  $U$  solves the equation  $U'' - rU' = 0$ . According to our assumptions,

$$r(z) = \frac{2\mu}{b} \frac{1}{x} + \frac{\varepsilon(x)}{x},$$

where  $\varepsilon(x) \rightarrow 0$  as  $x \rightarrow \infty$ . In view of the representation theorem for slowly varying functions, there exists a slowly varying at infinity function  $\ell(x)$  such that  $e^{R(x)} = x^{\rho-1}\ell(x)$  and  $U(x) \sim x^\rho \ell(x)/\rho$  where  $\rho = 2\mu/b + 1 > 0$ .

The main result in this section is the following theorem which provides exact asymptotics for stationary measure of recurrent Markov chains with asymptotically zero drift described above.

**Theorem 5.1.** *Let  $X_n$  be a recurrent Markov chain and let  $\pi(\cdot)$  be its stationary measure. Let  $\pi$  have right-unbounded support, that is,  $\pi(x, \infty) > 0$  for every  $x$ . Let (5.2)–(5.4) be valid and, for some increasing  $s(x) = o(x)$ ,*

$$\mathbb{P}\{\xi(x) < -s(x)\} = o(p(x)/x), \quad (5.8)$$

$$\mathbb{E}\{U(x + \xi(x)); \xi(x) > s(x)\} = o(p(x)/x)U(x), \quad (5.9)$$

$$\mathbb{E}\{|\xi(x)|^3; |\xi(x)| \leq s(x)\} = o(x^2 p(x)) \quad \text{as } x \rightarrow \infty. \quad (5.10)$$

Then

$$\pi(x_1, x_2] \sim c \int_{x_1}^{x_2} \frac{y}{U(y)} dy$$

as  $x_1, x_2 \rightarrow \infty$  in such a way that

$$1 < \liminf \frac{x_2}{x_1} \leq \limsup \frac{x_2}{x_1} < \infty.$$

**Corollary 5.2.** *If  $X_n$  is positive recurrent,  $2\mu > b$ , and conditions of Theorem 5.1 hold, then*

$$\pi(x, \infty) \sim \frac{c}{\rho - 2} \frac{x^2}{U(x)} \quad \text{as } x \rightarrow \infty.$$

*If  $X_n$  is null recurrent,  $2\mu \in (-b, b)$ , and conditions of Theorem 5.1 hold, then*

$$\pi(0, x) \sim \frac{c}{2 - \rho} \frac{x^2}{U(x)} \quad \text{as } x \rightarrow \infty.$$

**Corollary 5.3.** *Let, in addition,  $r(x) = 2\mu/bx$ . If  $2\mu > b$  and positive recurrence holds then*

$$\pi(x, \infty) \sim \frac{c\rho}{\rho - 2} \frac{1}{x^{2\mu/b-1}} \quad \text{as } x \rightarrow \infty.$$

*If  $2\mu \in (-b, b)$  and null recurrence holds then*

$$\pi(0, x) \sim \frac{c\rho}{2 - \rho} x^{1-2\mu/b} \quad \text{as } x \rightarrow \infty.$$

In paper [52], Menshikov and Popov investigated behaviour of the invariant distribution  $\{\pi(x), x \in \mathbb{Z}^+\}$  for countable Markov chains with asymptotically zero drift and with bounded jumps (see also Aspandiarov and Iasnogorodski [6]). Some rough theorems for the local probabilities  $\pi(x)$  were proved; if the condition (5.2) holds then for every  $\varepsilon > 0$  there exist constants  $c_- = c_-(\varepsilon) > 0$  and  $c_+ = c_+(\varepsilon) < \infty$  such that

$$c_- x^{-2\mu/b-\varepsilon} \leq \pi(x) \leq c_+ x^{-2\mu/b+\varepsilon}.$$

The paper [46] is devoted to the existence and non-existence of moments of invariant distribution. In particular, there was proven that if (5.2) holds and the families of random variables  $\{(\xi^+(x))^{2+\gamma}, x \geq 0\}$  for some  $\gamma > 0$  and  $\{(\xi^-(x))^2, x \geq 0\}$  are uniformly integrable then the moment of order  $\gamma$  of the invariant distribution  $\pi$  is finite if  $\gamma < 2\mu/b - 1$ , and infinite if  $\pi$  has unbounded support and  $\gamma > 2\mu/b - 1$ . This result implies that for every  $\varepsilon > 0$  there exists  $c(\varepsilon)$  such that

$$\pi(x, \infty) \leq c(\varepsilon) x^{-2\mu/b+1+\varepsilon}. \quad (5.11)$$

In [17] we have found the asymptotic behaviour of  $\pi(x, \infty)$  for positive recurrent chains under more restrictive moment conditions. In particular, it has been assumed that third moments of jumps converge.

Before proving Theorem 5.1 let us formulate and prove some auxiliary results. First we construct a Lyapunov function needed. Consider the function  $r_p(x) := r(x) - p(x)$  and define

$$\begin{aligned} R_p(x) &:= \int_0^x r_p(y) dy, \\ U_p(x) &:= \int_0^x e^{R_p(y)} dy. \end{aligned}$$

We have  $0 \leq r_p(x) \leq r(x)$ ,  $0 \leq R_p(x) \leq R(x)$  and  $0 \leq U_p(x) \leq U(x)$ . Since

$$C_p := \int_0^\infty p(y) dy \quad \text{is finite,}$$

we have

$$R_p(x) = R(x) - C_p + o(1) \quad \text{as } x \rightarrow \infty. \quad (5.12)$$

Therefore,

$$U_p(x) \sim e^{-C_p} U(x) \rightarrow \infty \quad \text{as } x \rightarrow \infty, \quad (5.13)$$



because  $U(x) \rightarrow \infty$ . Further, since  $xr_p(x) = xr(x) - xp(x) \rightarrow 2\mu/b$ ,

$$\frac{U'_p(x)}{(xe^{R_p(x)})'} = \frac{e^{R_p(x)}}{(1+xr_p(x))e^{R_p(x)}} \rightarrow \frac{b}{2\mu+b} \quad \text{as } x \rightarrow \infty.$$

Then L'Hospital's rule yields

$$U_p(x) \sim \frac{b}{2\mu+b} xe^{R_p(x)} \sim \frac{be^{-C_p}}{2\mu+b} xe^{R(x)} \quad \text{as } x \rightarrow \infty. \quad (5.14)$$

**Lemma 5.4.** *Assume that (5.8)–(5.10) hold. Then*

$$\mathbb{E}U_p(x + \xi(x)) - U_p(x) \sim -\frac{2\mu+b}{2} \frac{p(x)}{x} U_p(x) \quad \text{as } x \rightarrow \infty. \quad (5.15)$$

*Proof.* We start with the following decomposition:

$$\begin{aligned} \mathbb{E}U_p(x + \xi(x)) - U_p(x) &= \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) < -s(x)\} \\ &\quad + \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) > s(x)\}. \end{aligned} \quad (5.16)$$

Here the first term on the right is negative and may be bounded below as follows:

$$\begin{aligned} \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) < -s(x)\} &\geq -U_p(x) \mathbb{P}\{\xi(x) < -s(x)\} \\ &= o(p(x)/x) U_p(x), \end{aligned} \quad (5.17)$$

by the condition (5.8). Furthermore, the third term on the right hand side of (5.16) is positive and may be bounded above in the following way:

$$\begin{aligned} \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) > s(x)\} &\leq \mathbb{E}\{U_p(x + \xi(x)); \xi(x) > s(x)\} \\ &= o(p(x)/x) U_p(x), \end{aligned} \quad (5.18)$$

due to the condition (5.9). To estimate the second term on the right hand side of (5.16), we make use of Taylor's theorem:

$$\begin{aligned} \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); |\xi(x)| \leq s(x)\} &= U'_p(x) \mathbb{E}\{\xi(x); |\xi(x)| \leq s(x)\} + \frac{1}{2} U''_p(x) \mathbb{E}\{\xi^2(x); |\xi(x)| \leq s(x)\} \\ &\quad + \frac{1}{6} \mathbb{E}\{U'''_p(x + \theta\xi(x)) \xi^3(x); |\xi(x)| \leq s(x)\}, \end{aligned} \quad (5.19)$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ . By the construction of  $U_p$ ,

$$U'_p(x) = e^{R_p(x)}, \quad U''_p(x) = r_p(x) e^{R_p(x)} = (r(x) - p(x)) e^{R_p(x)}. \quad (5.20)$$

Then it follows that

$$\begin{aligned} U'_p(x) m_1^{[s(x)]}(x) + \frac{1}{2} U''_p(x) m_2^{[s(x)]}(x) &= e^{R_p(x)} \left( m_1^{[s(x)]}(x) + (r(x) - p(x)) \frac{m_2^{[s(x)]}(x)}{2} \right) \\ &= \frac{m_2^{[s(x)]}(x)}{2} e^{R_p(x)} \left( \frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} + r(x) - p(x) \right) \\ &= -\frac{m_2^{[s(x)]}(x)}{2} e^{R_p(x)} p(x) (1 + o(1)), \end{aligned}$$

by the condition (5.3). Hence, the equivalence (5.14) yields

$$U_p'(x)m_1^{[s(x)]}(x) + \frac{1}{2}U_p''(x)m_2^{[s(x)]}(x) \sim -m_2^{[s(x)]}(x)\frac{2\mu+b}{2b}\frac{p(x)}{x}U_p(x). \quad (5.21)$$

Finally, let us estimate the last term in (5.19). Notice that by the condition (5.4) on the derivatives of  $r(x)$  and  $p(x)$ ,

$$U_p'''(x) = (r'(x) - p'(x) + (r(x) - p(x))^2)e^{R_p(x)} = O(1/x^3)U_p(x).$$

so that

$$|\mathbb{E}\{U_p'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}| \leq \frac{c_1}{x^3}\mathbb{E}\{|\xi^3(x)|; |\xi(x)| \leq s(x)\}U_p(x).$$

Then, in view of (5.10),

$$|\mathbb{E}\{U_p'''(x + \theta\xi(x))\xi^3(x); |\xi(x)| \leq s(x)\}| = o(p(x)/x)U_p(x). \quad (5.22)$$

Then it follows from (5.19), (5.21) and (5.22) that

$$\begin{aligned} & \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); |\xi(x)| \leq s(x)\} \\ &= -m_2^{[s(x)]}(x)\frac{2\mu+b}{2b}\frac{p(x)}{x}U_p(x) + o(p(x)/x)U_p(x). \end{aligned} \quad (5.23)$$

Substituting (5.17), (5.18) and (5.23) into (5.16), we finally get the desired expression for  $\mathbb{E}U_p(x + \xi(x)) - U_p(x)$ . This completes the proof of the lemma.  $\square$

Fix an  $\hat{x} > 0$ . Define a transition kernel  $Q$  via the following change of measure

$$Q(x, dy) := \frac{U_p(y)}{U_p(x)}P(x, dy \cap (\hat{x}, \infty)).$$

We have

$$Q(x, \mathbb{R}^+) = \frac{\mathbb{E}\{U_p(x + \xi(x)); x + \xi(x) > \hat{x}\}}{U_p(x)}. \quad (5.24)$$

Lemma 5.4 yields the following result.

**Corollary 5.5.** *In conditions of Lemma 5.4, there exists an  $\hat{x}$  such that*

$$-(2\mu + b)\frac{p(x)}{x}U_p(x) \leq \mathbb{E}U_p(x + \xi(x)) - U_p(x) \leq 0$$

for all  $x > \hat{x}$  and

$$1 - (2\mu + b)\frac{p(x)}{x} \leq Q(x, \mathbb{R}^+) \leq 1.$$

Everywhere in what follows  $\widehat{x}$  is any level guaranteed by Corollary 5.5,  $B = (-\infty, \widehat{x}]$  and  $\tau_B := \min\{n \geq 1 : X_n \in B\}$ . Then definition of the transition kernel  $Q$  may be rewritten as follows

$$\begin{aligned} Q(x, dy) &= \frac{U_p(y)}{U_p(x)} \mathbb{P}_x\{X_1 \in dy, \tau_B > 1\} \\ &= \frac{U_p(y)}{U_p(x)} \mathbb{P}_x\{X_1 \in dy, X_1 > \widehat{x}\}, \quad x, y \geq 0, \end{aligned} \quad (5.25)$$

It follows from the upper bound in Corollary 5.5 that

$$Q(x, \mathbb{R}^+) = \frac{\mathbb{E}\{U_p(x + \xi(x)); \tau_B > 1\}}{U_p(x)} \leq \frac{\mathbb{E}U_p(x + \xi(x))}{U_p(x)} \leq 1 \quad \text{for all } x > \widehat{x}.$$

In other words,  $Q$  restricted to  $(\widehat{x}, \infty)$  is a substochastic kernel. It follows from (5.8) that

$$\begin{aligned} \mathbb{E}\{U_p(x + \xi(x)); \tau_B = 1\} &= \mathbb{E}_x\{U_p(X_1); X_1 \leq \widehat{x}\} \\ &\leq U_p(\widehat{x}) \mathbb{P}\{x + \xi(x) \leq \widehat{x}\} = o(p(x)/x). \end{aligned} \quad (5.26)$$

Combining it with the lower bound in Corollary 5.5 we obtain that

$$q(x) := -\log Q(x, \mathbb{R}^+) = O(p(x)/x). \quad (5.27)$$

Let us consider the following normalised kernel

$$\widehat{P}(x, dy) := \frac{Q(x, dy)}{Q(x, \mathbb{R}^+)}$$

and let  $\{\widehat{X}_n\}$  be a Markov chain with this transition probability; let  $\widehat{\xi}(x)$  be its jump from the state  $x$ . Consequently, performing the inverse change of measure we arrive at the following basic equality:

$$\begin{aligned} \mathbb{P}_x\{X_n \in dy, \tau_B > n\} &= \frac{U_p(x)}{U_p(y)} Q^n(x, dy) \\ &= \frac{U_p(x)}{U_p(y)} \mathbb{E}_x\{e^{-\sum_{k=0}^{n-1} q(\widehat{X}_k)}; \widehat{X}_n \in dy\}. \end{aligned} \quad (5.28)$$

**Lemma 5.6.** *Under the conditions of Lemma 5.4, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\widehat{\xi}(x); |\widehat{\xi}(x)| \leq s(x)\} \sim \frac{\mu + b}{x}, \quad (5.29)$$

$$\mathbb{E}\{(\widehat{\xi}(x))^2; |\widehat{\xi}(x)| \leq s(x)\} \rightarrow b, \quad (5.30)$$

$$\mathbb{P}\{|\widehat{\xi}(x)| > s(x)\} = o(p(x)/x). \quad (5.31)$$

Moreover, there exists a sufficiently large  $\widehat{x}$  such that

$$\mathbb{E}\{\widehat{\xi}(x); \widehat{\xi}(x) \leq s(x)\} \geq \frac{\mu + b}{2x} \quad \text{for all } x \geq \widehat{x}. \quad (5.32)$$

*Proof.* It follows from (5.12) and (5.14) that

$$\frac{U'_p(x)}{U_p(x)} = \frac{e^{R_p(x)}}{U_p(x)} \sim \frac{e^{R(x)-C_p}}{\frac{be^{-C_p}}{2\mu+b}xe^{R(x)}} = \frac{2\mu+b}{bx} = (1+2\mu/b)r(x)$$

where  $r(x) = 1/x$  and  $c_{12} = 2\mu/b$ . So, the function  $U_p$  satisfies the condition (4.11) with  $c_U = 1 + 2\mu/b$ . Also  $U_p$  satisfies (4.12) for any  $s(x) = o(x)$  because

$$\frac{U'_p(x+y)}{U'_p(x)} = \frac{e^{R_p(x+y)}}{e^{R_p(x)}} \sim e^{R(x+y)-R(x)} = e^{\int_x^{x+y} r(z)dz} = e^{O(s(x)/x)} = e^{o(1)}$$

as  $x \rightarrow \infty$  uniformly for all  $|y| \leq s(x)$ , and, by (5.14),

$$\frac{U_p(x+y)}{U_p(x)} \sim \frac{x+y}{x} \frac{e^{R(x+y)}}{e^{R(x)}} \sim e^{R(x+y)-R(x)} \rightarrow 1.$$

Finally,  $U_p$  satisfies (4.13) by Lemma 5.4. So, all conditions of Lemma 4.2 are met and (5.29)–(5.32) follow and the proof is complete.  $\square$

Therefore, the chain  $\widehat{X}_n$  satisfies the conditions (3.38) and (3.39) of Theorem 3.7 with  $\widehat{\mu} = \mu + b$  and  $\widehat{b} = b$ , so that  $\widehat{\mu} > \widehat{b}/2$ . Further, the lower bound (5.32) for the drift of  $\widehat{X}_n$  allows to apply Lemma 3.2 to  $\widehat{X}_n$  and to conclude that

$$\mathbb{E}_y \widehat{T}(t) = \mathbb{E}_y \widehat{L}(\widehat{x}, \widehat{T}(t)) < \infty \quad \text{for all } t > y,$$

so hence, for any initial state  $\widehat{X}_0 = y$ ,

$$\mathbb{P}\left\{\limsup_{n \rightarrow \infty} \widehat{X}_n = \infty\right\} = 1.$$

In its turn, then it follows from Theorem 2.18 that  $\widehat{X}_n \rightarrow \infty$  with probability 1.

So, Theorem 3.7 is applicable to  $\widehat{X}_n$  which implies weak convergence of  $(\widehat{X}_n)^2/n$  to  $\Gamma$ -distribution with mean  $2\mu + 3b$  and variance  $(2\mu + 3b)2b$ , that is,  $\Gamma$ -distribution with probability density function

$$\gamma(u) = \frac{1}{(2b)^{1+\rho/2}\Gamma(1+\rho/2)} u^{\rho/2} e^{-u/2b}. \quad (5.33)$$

Furthermore, by Theorem 3.3, there exists a  $c < \infty$  such that

$$\widehat{H}_y(x) := \sum_{n=0}^{\infty} \mathbb{P}_y\{\widehat{X}_n \leq x\} \leq c(1+x^2) \quad \text{for all } x, y > 0. \quad (5.34)$$

Having this estimate we now prove the following result.

**Lemma 5.7.** *Under the conditions of Lemma 5.4,*

$$h(z) := \lim_{n \rightarrow \infty} \mathbb{E}_z e^{-\sum_{k=0}^n q(\widehat{X}_k)} > 0 \quad \text{for all } z > 0,$$

where  $q$  is defined in (5.27). Moreover,  $h(z) \rightarrow 1$  as  $z \rightarrow \infty$ .

*Proof.* The existence of  $h(z)$  is immediate because  $e^{-\sum_{k=0}^n q(\hat{X}_k)}$  is decreasing in  $n$ . To show positivity it suffices to prove that

$$\mathbb{E}_z \sum_{k=1}^{\infty} q(\hat{X}_k) < \infty, \quad z > \hat{x}. \quad (5.35)$$

Note that

$$\mathbb{E}_z \sum_{k=1}^{\infty} q(\hat{X}_k) \leq \int_{\hat{x}}^{\infty} q(y) \hat{H}_z(dy) \leq c \int_{\hat{x}}^{\infty} \frac{p(y)}{y} \hat{H}_z(dy),$$

because  $q(y) = O(p(y)/y)$ , see (5.27). But it has been already shown in the proof of Theorem 3.3 that the last integral is finite.

To prove that  $h(z) \rightarrow 1$ , we note that Theorem 2.15 implies, for every fixed  $N > 0$ ,

$$\mathbb{P}_z\{\hat{X}_n > N \text{ for all } n \geq 1\} \rightarrow 1 \quad \text{as } z \rightarrow \infty,$$

so that

$$\hat{H}_z(N) \rightarrow 0 \quad \text{as } z \rightarrow \infty.$$

Then, for every fixed  $N$ ,

$$\lim_{z \rightarrow \infty} \mathbb{E}_z \sum_{k=0}^{\infty} q(\hat{X}_k) \leq \sup_{z > \hat{x}} \int_N^{\infty} q(y) \hat{H}_z(dy).$$

According to (3.30),

$$\lim_{N \rightarrow \infty} \sup_{z > \hat{x}} \int_N^{\infty} q(y) \hat{H}_z(dy) = 0.$$

Therefore, we infer that

$$\lim_{z \rightarrow \infty} \mathbb{E}_z \sum_{k=0}^{\infty} q(\hat{X}_k) = 0.$$

From this relation and Jensen inequality applied to the convex function  $e^{-z}$  we finally conclude  $\lim_{z \rightarrow \infty} h(z) = 1$ .  $\square$

**Corollary 5.8.** *Assume that the conditions of Lemma 5.4 are valid. Then  $h(x)$  is a harmonic function for the kernel  $Q$ , that is,*

$$h(x) = \int_{\hat{x}}^{\infty} h(y) Q(x, dy), \quad x > 0.$$

Furthermore,  $W(x) := h(x)U_p(x)$  is a harmonic function for  $X_n$  killed at the time of the first visit to  $[0, \hat{x}]$ :

$$W(x) = \mathbb{E}_x\{W(X_1); X_1 > \hat{x}\} \quad \text{for all } x > 0.$$

Consider the following weighted renewal measure

$$\widehat{H}_z^{(q)}(dx) = \sum_{n=0}^{\infty} \mathbb{E}_z \{ e^{-\sum_{k=0}^{n-1} q(\widehat{X}_k)}; \widehat{X}_n \in dx \}. \quad (5.36)$$

Combining Lemma 3.4, Lemma 5.7 and Theorem 3.8, we get

**Corollary 5.9.** *Assume that the conditions of Lemma 5.4 are valid. Then*

$$\widehat{H}_z^{(q)}(x_1, x_2] \sim h(z) \widehat{H}_z(x_1, x_2] \sim h(z) \frac{x_2^2 - x_1^2}{2\mu + b}$$

as  $x_1, x_2 \rightarrow \infty$  in such a way that

$$1 < \liminf \frac{x_2}{x_1} \leq \limsup \frac{x_2}{x_1} < \infty.$$

Now we are ready to prove the main result of this section.

*Proof of Theorem 5.1.* As follows from the representation (4.33) applied to  $U_p$ ,

$$\pi(x_1, x_2] = c^* \int_{x_1}^{x_2} \frac{H^{(q)}(dy)}{U_p(y)} \sim c^* e^{-C_p} \int_{x_1}^{x_2} \frac{H^{(q)}(dy)}{U(y)}, \quad (5.37)$$

due to  $U_p(y) \sim e^{C_p} U(y)$ , see (5.13);  $H^{(q)}$  is defined in (4.31). Integrating by parts, we obtain

$$\int_{x_1}^{x_2} \frac{\widehat{H}^{(q)}(dy)}{U(y)} = \frac{\widehat{H}^{(q)}(x_1, x_2]}{U(x_2)} - \int_{x_1}^{x_2} \widehat{H}^{(q)}(x_1, y] d \frac{1}{U(y)}.$$

Now, according to Corollary 5.9,

$$\widehat{H}^{(q)}(x_1, y] \sim c_q(y^2 - x_1^2),$$

where  $c_q := \mathbb{E}h(\widehat{X}_0)/(2\mu + b)$ . Consequently,

$$\int_{x_1}^{x_2} \frac{\widehat{H}^{(q)}(dy)}{U(y)} \sim c_q \frac{x_2^2 - x_1^2}{U(x_2)} - c_q \int_{x_1}^{x_2} (y^2 - x_1^2) d \frac{1}{U(y)}.$$

Integrating by parts once again we conclude the relation

$$\int_{x_1}^{x_2} \frac{\widehat{H}^{(q)}(dy)}{U(y)} \sim 2c_q \int_{x_1}^{x_2} \frac{y}{U(y)} dy,$$

which together with (5.37) concludes the proof.  $\square$

**Corollary 5.10.** *Assume that the conditions of Theorem 5.1 are valid. Then the integrability of  $y/U(y)$  is necessary and sufficient for the Markov chain  $X_n$  to be positive recurrent.*

## 5.2 Tail asymptotics for recurrence times of positive and null recurrent Markov chains

In this section we study the tail behaviour of the stopping time

$$\tau_{\hat{x}} := \inf\{n \geq 1 : X_n \leq \hat{x}\},$$

in the case where  $\tau_{\hat{x}}$  is a proper random variable, that is,  $X_n$  is either positive or null recurrent.

**Theorem 5.11.** *Let conditions of Theorem 5.1 hold. Then, for any fixed  $x > \hat{x}$ ,*

$$\mathbb{P}_x\{\tau_{\hat{x}} > n\} \sim \frac{1}{(2b)^{\rho/2}\Gamma(1+\rho/2)} \frac{W(x)}{U_p(\sqrt{n})} \quad \text{as } n \rightarrow \infty,$$

where  $W(x)$  is the harmonic function defined in Corollary 5.8.

Notice that

$$\frac{W(x)}{U_p(\sqrt{n})} \sim \frac{U(x)}{U(\sqrt{n})} \quad \text{as } n, x \rightarrow \infty,$$

due to Lemma 5.7 and the equivalence (5.13).

Alexander [4] has considered recurrence times for Markov chains with steps  $\pm 1$ . Using the standard embedding of such random walks into the corresponding Bessel process, he has found exact asymptotics for  $\mathbb{P}_x\{\tau_0 = n\}$  for all  $\rho > 0$ . Unfortunately, his method applies to skip-free chains only.

From the results in Hryniv et al. [34] one gets the bounds

$$n^{-\rho/2} \log^{-\varepsilon} n \leq \mathbb{P}_0\{\tau_0 > n\} \leq n^{-\rho/2} \log^{\rho+1+\varepsilon}$$

for chains satisfying conditions of Theorem 5.11 with  $r(x) = 2\mu/bx + o(1/x \log x)$ .

For the purpose to prove the last theorem we need the following upper bound for the tail of  $\tau$  which is precise up to a constant multiplier.

**Lemma 5.12.** *In conditions of Theorem 5.11, there exists a constant  $c < \infty$  such that*

$$\mathbb{P}_x\{\tau_{\hat{x}} > n\} \leq c \frac{U(x)}{U(\sqrt{n})} \quad \text{for all } n \text{ and } x > \hat{x}.$$

In its turn, in order to prove the last lemma we need a couple of preliminary results. In Lemma 3.1 we have already constructed a function of a transient Markov chain which is a bounded supermartingale. It turns out that for the Markov chain  $\hat{X}_n$  which is specially constructed possesses a similar result is valid under weaker conditions on the left tail distribution.

**Lemma 5.13.** *For any  $\varepsilon \in (0, 1)$  and  $a > 1/\varepsilon$ , there exists an  $x_* > \hat{x}$  such that*

$$\min\left(\frac{U_{ap}^\varepsilon(\hat{X}_n)}{U_p(\hat{X}_n)}, \frac{U_{ap}^\varepsilon(x_*)}{U_p(x_*)}\right)$$

is a positive supermartingale.

Then we can apply the Doob inequality which guarantees that there exists a constant  $c_1(\varepsilon)$  such that

$$\mathbb{P}_z\{\widehat{X}_k \leq y \text{ for some } k \geq 1\} \leq c_1(\varepsilon) \frac{U_p(y)}{U_p(z)} \frac{U_{ap}^{\varepsilon/2\rho}(z)}{U_{ap}^{\varepsilon/2\rho}(y)} \text{ for all } y < z.$$

Hence the equivalence (5.13) implies existence of  $c_2(\varepsilon)$  such that

$$\mathbb{P}_z\{\widehat{X}_k \leq y \text{ for some } k \geq 1\} \leq c_2(\varepsilon) \left( \frac{U(y)}{U(z)} \right)^{1-\varepsilon/2\rho} \text{ for all } y < z.$$

Since  $U$  is regularly varying with index  $\rho$ , there exists a constant  $c_3(\varepsilon)$  such that

$$\frac{1}{c_3(\varepsilon)} \left( \frac{y}{z} \right)^{\rho+\varepsilon/2} \leq \frac{U(y)}{U(z)} \leq c_3(\varepsilon) \left( \frac{y}{z} \right)^{\rho-\varepsilon/2} \text{ for all } y < z. \quad (5.38)$$

Consequently,

$$\mathbb{P}_z\{X_k \leq y \text{ for some } k \geq 1\} \leq c_4(\varepsilon) \left( \frac{y}{z} \right)^{\rho-\varepsilon}. \quad (5.39)$$

*Proof of Lemma 5.13.* By the definition of the chain  $\widehat{X}_n$  and Jensen's inequality,

$$\begin{aligned} \mathbb{E} \frac{U_{ap}^\varepsilon(x + \widehat{\xi}(x))}{U_p(x + \widehat{\xi}(x))} &= \int_{\widehat{x}}^\infty \frac{U_{ap}^\varepsilon(y)}{U_p(y)} \frac{Q(x, dy)}{Q(x, \mathbb{R}^+)} \\ &= \frac{1}{\int_{\widehat{x}}^\infty U_p(y) P(x, dy)} \int_{\widehat{x}}^\infty U_{ap}^\varepsilon(y) P(x, dy) \\ &\leq \frac{1}{\int_{\widehat{x}}^\infty U_p(y) P(x, dy)} \left( \int_{\widehat{x}}^\infty U_{ap}(y) P(x, dy) \right)^\varepsilon. \end{aligned} \quad (5.40)$$

Due to Lemma 5.4 and (5.26), as  $x \rightarrow \infty$ ,

$$\int_{\widehat{x}}^\infty U_p(y) P(x, dy) = U_p(x) \left( 1 - \frac{2\mu + b}{2} \frac{p(x)}{x} + o\left(\frac{p(x)}{x}\right) \right) \quad (5.41)$$

and

$$\int_{\widehat{x}}^\infty U_{ap}(y) P(x, dy) = U_{ap}(x) \left( 1 - a \frac{2\mu + b}{2} \frac{p(x)}{x} + o\left(\frac{p(x)}{x}\right) \right).$$

Then

$$\left( \int_{\widehat{x}}^\infty U_{ap}(y) P(x, dy) \right)^\varepsilon = U_{ap}^\varepsilon(x) \left( 1 - a\varepsilon \frac{2\mu + b}{2} \frac{p(x)}{x} + o\left(\frac{p(x)}{x}\right) \right)$$

and it follows from  $a\varepsilon > 1$  that

$$\frac{1}{\int_{\widehat{x}}^\infty U_p(y) P(x, dy)} \left( \int_{\widehat{x}}^\infty U_{ap}(y) P(x, dy) \right)^\varepsilon \leq \frac{U_{ap}^\varepsilon(x)}{U_p(x)}.$$

for all sufficiently large  $x$  which completes the proof.  $\square$



**Lemma 5.14.** *For*

$$\widehat{T}(z) := \min\{n \geq 0 : \widehat{X}_n \geq z\},$$

*there exists a  $\gamma > 0$  such that, for all  $n$  and  $z$ ,*

$$\sup_x \mathbb{P}_x\{T(z) > n\} \leq c_4 e^{-\gamma n/z^2}.$$

*Proof.* It follows from the definition of the chain  $\widehat{X}_n$  that it only visits  $[0, \widehat{x}]$  at time 0. Therefore,

$$\widehat{T}(z) \leq 1 + \sum_{k=1}^{\widehat{T}(z)-1} \mathbb{I}\{\widehat{X}_k > \widehat{x}\}.$$

Then, by Lemma 3.2,

$$\mathbb{E}_x \widehat{T}(z) \leq c_3 z^2 \quad \text{uniformly for all } x \text{ and } z. \quad (5.42)$$

Next, by the Markov property, for every  $t$  and  $s > 0$ ,

$$\begin{aligned} \mathbb{P}_x\{\widehat{T}(z) > t + s\} &= \int_0^z \mathbb{P}_x\{\widehat{T}(z) > t, X_t \in du\} \mathbb{P}_u\{\widehat{T}(z) > s\} \\ &\leq \mathbb{P}_x\{\widehat{T}(z) > t\} \sup_{u \leq z} \mathbb{P}_u\{\widehat{T}(z) > s\}. \end{aligned}$$

Therefore, the monotone function  $g(t) := \sup_{u \leq z} \mathbb{P}_u\{\widehat{T}(z) > tz^2\}$  satisfies the inequality  $g(t+s) \leq g(t)g(s)$ . Then the increasing function  $g_0(t) := \log(1/g(t))$  is convex and  $g_0(0) = 0$ . By the bound (5.42) and Markov's inequality, there exists  $t_0$  such that  $g(t_0) < 1$  so that  $g(t_0) = e^{-\gamma}$  with  $\gamma > 0$ , and  $g_0(t_0) = \gamma > 0$ . Then, by  $g_0(0) = 0$  and convexity of  $g_0$ ,  $g_0(t) \geq \gamma(t - t_0)$  which implies  $g(t) \leq e^{-\gamma(t-t_0)}$  equivalent to the lemma conclusion.  $\square$

**Lemma 5.15.** *For any fixed  $\varepsilon > 0$ , there exists a constant  $c_6 = c_6(\varepsilon)$  such that, for all  $n$ ,  $x$  and  $y \leq \sqrt{n}$ ,*

$$\mathbb{P}_x\{\widehat{X}_k \leq y \text{ for some } k \in [n+1, 2n]\} \leq c_6 \left(\frac{y}{\sqrt{n}}\right)^{\rho-\varepsilon}.$$

*Proof.* For any  $z > y$ , the event whose probability we need to bound from above can only happen if either the chain  $\widehat{X}$  does not exceed the level  $z$  within time  $n$  or it does exceed this level and then falls down below  $y$ , so, by the Markov property, the corresponding probability is not greater than the sum

$$\mathbb{P}_x\{T(z) > n\} + \sup_{u \geq z} \mathbb{P}_u\{\widehat{X}_k \leq y \text{ for some } k \geq 1\},$$

where the first term may be bounded above by Lemma 5.14 and the second term by (5.39), so

$$\mathbb{P}_x\{\widehat{X}_k \leq y \text{ for some } k \in [n+1, 2n]\} \leq c_5 \left( e^{-\gamma n/z^2} + \left(\frac{y}{z}\right)^{\rho-\varepsilon} \right).$$

Optimisation of the right hand side with respect to  $z$  is not solvable in elementary functions, so we choose

$$z := \sqrt{\frac{\gamma n}{\log((\sqrt{n}/y)^{\rho-\varepsilon})}},$$

which is close to the optimal value. Then

$$\begin{aligned} \mathbb{P}_x\{\widehat{X}_k \leq y \text{ for some } k \in [n+1, 2n]\} \\ \leq c_5 \left( \left( \frac{y}{\sqrt{n}} \right)^{\rho-\varepsilon} + \left( \frac{y}{\sqrt{\gamma n}} \right)^{\rho-\varepsilon} \left( (\rho-\varepsilon) \log \frac{\sqrt{n}}{y} \right)^{\frac{\rho-\varepsilon}{2}} \right), \end{aligned}$$

which implies the lemma conclusion if we take  $\varepsilon/2$  instead of  $\varepsilon$  on the right hand side.  $\square$

*Proof of Lemma 5.12.* It follows from (5.28) that

$$\begin{aligned} \mathbb{P}_x\{\tau_{\widehat{x}} > n\} &= U_p(x) \int_{\widehat{x}}^{\infty} \frac{1}{U_p(y)} Q^n(x, dy) \\ &= U_p(x) \int_{\widehat{x}}^{\infty} \frac{1}{U_p(y)} \mathbb{E}_x\{e^{-\sum_{k=0}^{n-1} q(\widehat{X}_k)}; \widehat{X}_n \in dy\}. \end{aligned} \quad (5.43)$$

Since  $Q$  is substochastic,

$$\begin{aligned} \mathbb{P}_x\{\tau_{\widehat{x}} > n\} &\leq U_p(x) \mathbb{E}_x \frac{1}{U_p(\widehat{X}_n)} \\ &\leq c_1 U(x) \mathbb{E}_x \frac{1}{U(\widehat{X}_n)}, \end{aligned} \quad (5.44)$$

due to (5.13). Summing up  $n$  successive probabilities we get

$$\begin{aligned} \sum_{k=n+1}^{2n} \mathbb{P}_x\{\tau_{\widehat{x}} > k\} &\leq c_1 U(x) \int_{\widehat{x}}^{\infty} \frac{1}{U(y)} \widehat{H}_{x,n}(dy) \\ &= c_1 U(x) \left( \int_{\widehat{x}}^{\sqrt{n}} + \int_{\sqrt{n}}^{\infty} \right) \frac{1}{U(y)} \widehat{H}_{x,n}(dy), \end{aligned} \quad (5.45)$$

where

$$\widehat{H}_{x,n}(A) := \sum_{k=n+1}^{2n} \mathbb{P}_x\{\widehat{X}_k \in A\}.$$

The function  $U$  increases, so

$$\int_{\sqrt{n}}^{\infty} \frac{1}{U(y)} \widehat{H}_{x,n}(dy) \leq \frac{n}{U(\sqrt{n})} \quad \text{for all } x \text{ and } n. \quad (5.46)$$

Further, integrating by parts, we obtain

$$\begin{aligned} \int_{\hat{x}}^{\sqrt{n}} \frac{1}{U(y)} \hat{H}_{x,n}(dy) &\leq \frac{\hat{H}_{x,n}(\hat{x}, \sqrt{n})}{U(\sqrt{n})} + \int_{\hat{x}}^{\sqrt{n}} \frac{U'(y) \hat{H}_{x,n}(\hat{x}, y]}{U^2(y)} dy \\ &\leq \frac{n}{U(\sqrt{n})} + \int_{\hat{x}}^{\sqrt{n}} \frac{e^{R(y)} \hat{H}_{x,n}(\hat{x}, y]}{U^2(y)} dy. \end{aligned}$$

Combining this with (5.45), (5.46) and noting that  $e^{R(y)} \sim \rho \frac{U(y)}{y}$ , we conclude that

$$\sum_{k=n+1}^{2n} \mathbb{P}_x\{\tau_{\hat{x}} > k\} \leq 2c_1 U(x) \frac{n}{U(\sqrt{n})} + c_2 U(x) \int_{\hat{x}}^{\sqrt{n}} \frac{\hat{H}_{x,n}(\hat{x}, y]}{y U(y)} dy, \quad (5.47)$$

with some constant  $c_2$  which does not depend on  $x$ .

We next derive an upper bound for  $\hat{H}_{x,n}$ . It is clear that

$$\begin{aligned} \hat{H}_{x,n}(\hat{x}, y] &= \mathbb{E}_x \sum_{k=n+1}^{2n} \mathbb{I}\{\hat{X}_k \in (\hat{x}, y]\} \\ &\leq \mathbb{P}_x\{\hat{X}_k \in (\hat{x}, y] \text{ for some } k \in [n+1, 2n]\} \sup_{s \leq y} \sum_{k=0}^{\infty} \mathbb{P}_s\{\hat{X}_k \in (\hat{x}, y]\} \\ &\leq \sup_s \hat{H}_s(\hat{x}, y] \mathbb{P}_x\{\hat{X}_k \in (\hat{x}, y] \text{ for some } k \in [n+1, 2n]\}. \end{aligned}$$

Applying here Theorem 3.3 and Lemma 5.15, we get

$$\hat{H}_{x,n}(\hat{x}, y] \leq c_3 y^2 \left( \frac{y}{\sqrt{n}} \right)^{\rho-\varepsilon}.$$

Therefore,

$$\int_{\hat{x}}^{\sqrt{n}} \frac{\hat{H}_{x,n}(y)}{y U(y)} dy \leq c_4 \int_{\hat{x}}^{\sqrt{n}} \frac{y}{U(y)} \left( \frac{y}{\sqrt{n}} \right)^{\rho-\varepsilon} dy.$$

Substitution  $y = u\sqrt{n}$  leads to the following expression for the last integral:

$$\frac{n}{U(\sqrt{n})} \int_{\hat{x}/\sqrt{n}}^1 \frac{U(\sqrt{n})}{U(u\sqrt{n})} u^{1+\rho-\varepsilon} du.$$

Applying the left hand side inequality in (5.38) we get an upper bound

$$\int_{\hat{x}}^{\sqrt{n}} \frac{\hat{H}_{x,n}(y)}{y U(y)} dy \leq c_5 \frac{n}{U(\sqrt{n})} \int_0^1 u^{1-2\varepsilon} du = c_6 \frac{n}{U(\sqrt{n})},$$

provided  $\varepsilon < 1$ . Substituting this upper bound into (5.47) we get that

$$\sum_{k=n+1}^{2n} \mathbb{P}_x\{\tau_{\hat{x}} > k\} \leq C U(x) \frac{n}{U(\sqrt{n})}.$$

Therefore,

$$\mathbb{P}_x\{\tau_{\hat{x}} > 2n\} \leq C \frac{U(x)}{U(\sqrt{n})}.$$

Since  $U$  is regularly varying at infinity, this completes the proof.  $\square$

*Proof of Theorem 5.11.* Fix an  $\varepsilon > 0$  and split the integral (5.43) into two parts

$$\mathbb{P}_x\{\tau_{\hat{x}} > n\} = U_p(x) \left( \int_{\hat{x}}^{\varepsilon\sqrt{n}} + \int_{\varepsilon\sqrt{n}}^{\infty} \right) \frac{1}{U_p(y)} Q^n(x, dy). \quad (5.48)$$

The asymptotic behaviour of the second integral on the right hand side relatively easy follows from the weak convergence to  $\Gamma$ -distribution and dominated convergence theorem. Indeed,

$$\int_{\varepsilon\sqrt{n}}^{\infty} \frac{1}{U_p(y)} Q^n(x, dy) = \frac{1}{U_p(\sqrt{n})} \int_{\varepsilon\sqrt{n}}^{\infty} \frac{U_p(\sqrt{n})}{U_p(y)} Q^n(x, dy). \quad (5.49)$$

Monotonicity of  $U_p$  implies the following upper bound for the integrand on the right hand side:

$$\sup_{n, y > \varepsilon\sqrt{n}} \frac{U_p(\sqrt{n})}{U_p(y)} \leq \sup_n \frac{U_p(\sqrt{n})}{U_p(\varepsilon\sqrt{n})} < \infty, \quad (5.50)$$

because  $U_p$  is regularly varying at infinity which also implies convergence

$$\frac{U_p(\sqrt{n})}{U_p(u\sqrt{n})} \rightarrow \frac{1}{u^\rho} \quad \text{as } n \rightarrow \infty. \quad (5.51)$$

It follows from Theorem 3.7 that  $\hat{X}_n^2/n$  converges weakly to the  $\Gamma$ -distribution with probability density function  $\gamma(u)$ , see (5.33). Then, by Lemma 3.5, the substochastic measure  $Q^n(x, \sqrt{n} \cdot du)$  converges weakly as  $n \rightarrow \infty$  to a measure with density function  $h(x)2u\gamma(u^2)$ . The relations (5.50) and (5.51) allow to apply the dominated convergence theorem and to conclude that

$$\begin{aligned} \int_{\varepsilon}^{\infty} \frac{U_p(\sqrt{n})}{U_p(u\sqrt{n})} Q^n(x, \sqrt{n} \cdot du) &\rightarrow h(x) \int_{\varepsilon}^{\infty} \frac{2u}{u^\rho} \gamma(u^2) du \\ &\rightarrow h(x) \int_{\varepsilon^2}^{\infty} \frac{1}{u^{\rho/2}} \gamma(u) du \\ &= h(x) \frac{e^{-\varepsilon^2/2b}}{(2b)^{\rho/2} \Gamma(1 + \rho/2)}. \end{aligned}$$

Hence, (5.49) finally implies

$$\begin{aligned} U_p(x) \int_{\varepsilon\sqrt{n}}^{\infty} \frac{1}{U_p(y)} Q^n(x, dy) &\sim \frac{h(x)U_p(x)}{U_p(\sqrt{n})} \frac{e^{-\varepsilon^2/2b}}{(2b)^{\rho/2} \Gamma(1 + \rho/2)} \\ &= \frac{W(x)}{U_p(\sqrt{n})} \frac{e^{-\varepsilon^2/2b}}{(2b)^{\rho/2} \Gamma(1 + \rho/2)}. \end{aligned} \quad (5.52)$$

Letting  $\varepsilon \downarrow 0$  we conclude the following lower bound

$$\liminf_{n \rightarrow \infty} U_p(\sqrt{n}) \mathbb{P}_x\{\tau_{\hat{x}} > n\} \geq \frac{W(x)}{(2b)^{\rho/2} \Gamma(1 + \rho/2)}. \quad (5.53)$$

Fix some  $\delta > 0$ . By the Markov property,

$$\mathbb{P}_x\{\tau_{\hat{x}} > n\} = \int_{\hat{x}}^{\infty} \mathbb{P}_x\{X_{(1-\delta)n} \in dy, \tau_{\hat{x}} > (1-\delta)n\} \mathbb{P}_y\{\tau_{\hat{x}} > \delta n\}. \quad (5.54)$$

It follows from Lemma 5.12 that

$$\begin{aligned} & \int_{\hat{x}}^{\varepsilon\sqrt{n}} \mathbb{P}_x\{X_{(1-\delta)n} \in dy, \tau_{\hat{x}} > (1-\delta)n\} \mathbb{P}_y\{\tau_{\hat{x}} > \delta n\} \\ & \leq \frac{C}{U_p(\sqrt{\delta n})} \int_{\hat{x}}^{\varepsilon\sqrt{n}} U_p(y) \mathbb{P}_x\{X_{(1-\delta)n} \in dy, \tau_{\hat{x}} > (1-\delta)n\} \\ & = \frac{CU_p(x)}{U_p(\sqrt{\delta n})} \int_{\hat{x}}^{\varepsilon\sqrt{n}} Q^{(1-\delta)n}(x, dy) \\ & \leq \frac{CU_p(x)}{U_p(\sqrt{\delta n})} \mathbb{P}_x\{\hat{X}_{(1-\delta)n} \leq \varepsilon\sqrt{n}\}, \end{aligned}$$

since  $Q$  is substochastic. The function  $U_p$  is regularly varying with index  $\rho$ , so hence  $U_p(\sqrt{\delta n})/U_p(\sqrt{n}) \rightarrow \delta^{\rho/2}$  as  $n \rightarrow \infty$ . Together with weak convergence of  $\hat{X}_n^2/n$  to  $\Gamma$ -distribution, it implies that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} U_p(\sqrt{n}) \int_{\hat{x}}^{\varepsilon\sqrt{n}} \mathbb{P}_x\{X_{(1-\delta)n} \in dy, \tau_{\hat{x}} > (1-\delta)n\} \mathbb{P}_y\{\tau_{\hat{x}} > \delta n\} = 0. \quad (5.55)$$

Further,

$$\begin{aligned} & \int_{\varepsilon\sqrt{n}}^{\infty} \mathbb{P}_x\{X_{(1-\delta)n} \in dy, \tau_{\hat{x}} > (1-\delta)n\} \mathbb{P}_y\{\tau_{\hat{x}} > \delta n\} \\ & \leq \int_{\varepsilon\sqrt{n}}^{\infty} \mathbb{P}_x\{X_{(1-\delta)n} \in dy, \tau_{\hat{x}} > (1-\delta)n\} \\ & = U_p(x) \int_{\varepsilon\sqrt{n}}^{\infty} \frac{1}{U_p(y)} Q^{(1-\delta)n}(x, dy). \end{aligned}$$

As proven in (5.52),

$$U_p(x) \int_{\varepsilon\sqrt{n}}^{\infty} \frac{1}{U_p(y)} Q^{(1-\delta)n}(x, dy) \sim \frac{W(x)}{U_p(\sqrt{(1-\delta)n})} \frac{e^{-\varepsilon^2/2b(1-\delta)}}{(2b)^{\rho/2} \Gamma(1 + \rho/2)}. \quad (5.56)$$

Substitution of (5.55) and (5.56) into (5.54) leads to

$$\limsup_{n \rightarrow \infty} U_p(\sqrt{n}) \mathbb{P}_x\{\tau_{\hat{x}} > n\} \leq \limsup_{n \rightarrow \infty} \frac{W(x) U_p(\sqrt{n})}{U_p(\sqrt{(1-\delta)n})} \frac{1}{(2b)^{\rho/2} \Gamma(1 + \rho/2)}.$$

Since  $U_p(\sqrt{n})/U_p(\sqrt{(1-\delta)n}) \rightarrow (1-\delta)^{-\rho/2}$  and  $\delta > 0$  may be chosen as small as we please, we obtain the upper bound

$$\limsup_{n \rightarrow \infty} U_p(\sqrt{n}) \mathbb{P}_x \{\tau_{\hat{x}} > n\} \leq \frac{W(x)}{(2b)^{\rho/2} \Gamma(1 + \rho/2)},$$

which together with lower bound (5.53) completes the proof.  $\square$

**Corollary 5.16.** *Assume that  $X_n$  is a countable Markov chain on a state space  $\{z_0 < z_1 < z_2 < \dots\}$ . Assume also that  $\mathbb{E}U(\xi(z_k)) < \infty$  for all  $k \geq 0$ . Then, for any fixed states  $x$  and  $z$  there exists a constant  $C(x, z)$  such that*

$$\mathbb{P}_x \{\sigma_z > n\} \sim \frac{C(x, z)}{U(\sqrt{n})} \text{ as } n \rightarrow \infty,$$

where  $\sigma_z := \min\{n \geq 1 : X_n = z\}$ .

*Proof.* Since there are finitely many states in the compact  $[0, \hat{x}]$ ,  $\sigma_z$  can be represented as a random sum—with a geometric number of terms at the most—of recurrence times to the compact  $B = [0, \hat{x}]$ . According to Theorem 5.11, all these recurrence times have the same, up to a constant factor, regularly varying at infinity tail behaviour which is known to be of subexponential type. Thus, arguments based on the Markov property and Kesten's bound—see e.g. [26, Sec. 3.10]—show that the random sum follows the same regularly varying tail asymptotics.  $\square$

### 5.3 Limit theorems for conditioned positive and null recurrent Markov chains

In this section we prove limit theorems for positive and null recurrent Markov chains  $X_n$  conditioned on the event

$$\{X_1 > \hat{x}, \dots, X_n > \hat{x}\}.$$

**Theorem 5.17.** *Let conditions of Theorem 5.1 hold. Then*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{X_n^2}{n} > u \mid \tau_{\hat{x}} > n \right\} = e^{-u/2b} \text{ for all } u > 0.$$

*Proof.* For any fixed initial state  $x$ , by the change of measure,

$$\begin{aligned} \mathbb{P}_x \left\{ \frac{X_n^2}{n} > u, \tau_{\hat{x}} > n \right\} &= U_p(x) \int_{\sqrt{un}}^{\infty} \frac{1}{U_p(y)} Q_n(x, dy) \\ &\sim \frac{W(x)}{U_p(\sqrt{n})} \frac{e^{-u/2b}}{(2b)^{\rho/2} \Gamma(1 + \rho/2)}. \end{aligned}$$

as shown in (5.52). Combining this with tail asymptotics for  $\tau_{\hat{x}}$  given in Theorem 5.11, we arrive at the required result.  $\square$

**Corollary 5.18.** *Assume that  $X_n$  is a countable Markov chain on a state space  $\{z_0 < z_1 < z_2 < \dots\}$ . Then, for any fixed state  $z$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left\{\frac{X_n^2}{n} > u \mid \sigma_z > n\right\} = e^{-u/2b}, \quad u > 0,$$

where  $\sigma_z := \min\{n \geq 1 : X_n = z\}$ .

*Proof.* Fix some initial state  $x$  and  $N \geq 1$ . Then, by the Markov property,

$$\mathbb{P}_x\{\sigma_z > n, X_k \leq \hat{x} \text{ for some } k \in [N, n/2]\} \leq \mathbb{P}_x\{\sigma_z > N\} \max_{y \leq \hat{x}} \mathbb{P}_y\{\sigma_z > n/2\}$$

and

$$\mathbb{P}_x\{\sigma_z > n, X_j \leq \hat{x} \text{ for some } k \in [n/2, n - N]\} \leq \mathbb{P}_x\{\sigma_z > n/2\} \max_{y \leq \hat{x}} \mathbb{P}_y\{\sigma_z > N\}.$$

From these inequalities and Corollary 5.16 we infer that

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\mathbb{P}_x\{\sigma_z > n, X_j \leq \hat{x} \text{ for some } k \in [N, n - N]\}}{\mathbb{P}_x\{\sigma_z > n\}} = 0 \quad (5.57)$$

Further, since the tail of  $\tau_B$  is regularly varying,  $\mathbb{P}\{\tau_B = n + k\} = o(\mathbb{P}\{\tau_B > n\})$  as  $n \rightarrow \infty$  for any fixed  $k \in \mathbb{Z}$ , so

$$\begin{aligned} & \mathbb{P}_x\{\sigma_z > n, X_j > \hat{x} \text{ for all } k \in [N, n - N], X_j \leq \hat{x} \text{ for some } j \in (n - N, n]\} \\ & \leq \mathbb{P}_{X_N}\{X_j > \hat{x} \text{ for all } k \in [N, n - N], X_j \leq \hat{x} \text{ for some } j \in (n - N, n]\} \\ & = \mathbb{P}_{X_N}\{\tau_B \in [n - 2N, n - N]\} \\ & = o(\mathbb{P}_{X_N}\{\tau_B > n\}) = o(\mathbb{P}_x\{\sigma_z > n\}). \end{aligned} \quad (5.58)$$

Further, by the Markov property at the time of last visit to  $[0, \hat{x}]$ ,

$$\begin{aligned} & \mathbb{P}_x\left\{\frac{X_n^2}{n} > u, \sigma_z > n, X_j > \hat{x} \text{ for all } j \in [N, n]\right\} \\ & = \sum_{j=0}^{N-1} \int_B \mathbb{P}_x\{X_j \in dy, \sigma_z > j\} \mathbb{P}_y\left\{\frac{X_n^2}{n} > u, \tau_B > n - j\right\}. \end{aligned}$$

Applying now Theorem 5.17, we get

$$\begin{aligned} & \mathbb{P}_x\left\{\frac{X_n^2}{n} > u, \sigma_z > n, X_j > \hat{x} \text{ for all } j \in [N, n]\right\} \\ & \sim e^{-u/2b} \mathbb{P}_x\{\sigma_z > n, X_j > \hat{x} \text{ for all } j \in [N, n]\}. \end{aligned} \quad (5.59)$$

Combining (5.57)–(5.59) we arrive at required limit behaviour.  $\square$

**Theorem 5.19.** *Let conditions of Theorem 5.1 hold. Then*

$$\mathbb{P}_x\left\{\max_{n \leq \tau_{\hat{x}}} X_n > y\right\} \sim \frac{W(x)}{W(y)} \quad \text{as } y \rightarrow \infty,$$

where  $W$  is the harmonic function for  $X_n$  killed at the time of the first visit to  $[0, \hat{x}]$ , see Corollary 5.8.

This result improves Theorem 2.3 by Hryniv et al. [34] where lower and upper bounds were given with extra logarithmic term.

*Proof.* First notice that

$$\mathbb{P}_x \left\{ \max_{n \leq \tau_{\hat{x}}} X_n > y \right\} = \mathbb{P}_x \{ \tau_{\hat{x}} > T(y) \}.$$

Harmonicity of  $W$  implies that the sequence  $W(X_n) \mathbb{I}\{\tau_{\hat{x}} > n\}$  is a martingale. Applying the optional stopping theorem to this martingale and to the stopping time  $\tau_{\hat{x}} \wedge T(y)$ , we have

$$W(x) = \mathbb{E}_x \{ W(X_{T(y)}); \tau_{\hat{x}} > T(y) \}.$$

Since  $W(z) \sim U_p(z)$  as  $z \rightarrow \infty$ , we have

$$\mathbb{E}_x \{ U_p(X_{T(y)}); \tau_{\hat{x}} > T(y) \} \rightarrow W(x) \quad \text{as } y \rightarrow \infty. \quad (5.60)$$

Let us split the expectation on the right hand side into two parts:

$$\begin{aligned} \mathbb{E}_x \{ U_p(X_{T(y)}); \tau_{\hat{x}} > T(y) \} &= \mathbb{E}_x \{ U_p(X_{T(y)}); \tau_{\hat{x}} > T(y), X_{T(y)} \leq y + s(y) \} \\ &\quad + \mathbb{E}_x \{ U_p(X_{T(y)}); \tau_{\hat{x}} > T(y), X_{T(y)} > y + s(y) \}. \end{aligned} \quad (5.61)$$

Since  $s(y) = o(y)$  and  $U_p$  is a regularly varying function,  $U_p(y + s(y)) \sim U_p(y)$  as  $y \rightarrow \infty$ , so

$$\begin{aligned} \mathbb{E}_x \{ U_p(X_{T(y)}); \tau_{\hat{x}} > T(y), X_{T(y)} \leq y + s(y) \} \\ \sim U_p(y) \mathbb{P}_x \{ \tau_{\hat{x}} > T(y), X_{T(y)} \leq y + s(y) \}. \end{aligned} \quad (5.62)$$

By the change of measure with function  $U_p$  and the fact that the resulting kernel  $Q$  is substochastic,

$$\mathbb{E}_x \{ U_p(X_{T(y)}), \tau_{\hat{x}} > T(y), X_{T(y)} > y + s(y) \} \leq U_p(x) \mathbb{P}_x \{ \hat{X}_{\hat{T}(y)} > y + s(y) \}.$$

By the formula of total probability,

$$\begin{aligned} \mathbb{P}_x \{ \hat{X}_{\hat{T}(y)} > y + s(y) \} &= \sum_{n=0}^{\infty} \int_{\hat{x}}^y \mathbb{P}_x \{ \hat{X}_n \in dz, \hat{T}(y) > n \} \mathbb{P} \{ \hat{\xi}(z) > y + s(y) - z \} \\ &\leq \int_{\hat{x}}^y \mathbb{P} \{ \hat{\xi}(z) > s(y) \} \hat{H}_x(dz) \end{aligned}$$

According to (5.31),  $\mathbb{P} \{ \hat{\xi}(z) > s(z) \} = o(p(z)/z)$ . Then

$$\int_{\hat{x}}^{\infty} \mathbb{P} \{ \hat{\xi}(z) > s(z) \} \hat{H}_x(dz) < \infty.$$

Consequently,

$$\int_{\hat{x}}^y \mathbb{P} \{ \hat{\xi}(z) > s(y) \} \hat{H}_x(dz) \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$



As a result,

$$\mathbb{E}_x\{U_p(X_{T(y)}); \tau_{\hat{x}} > T(y), X_{T(y)} > y + s(y)\} \rightarrow 0 \quad \text{as } y \rightarrow \infty, \quad (5.63)$$

and hence

$$U_p(y)\mathbb{P}_x\{\tau_{\hat{x}} > T(y), X_{T(y)} > y + s(y)\} \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (5.64)$$

Applying (5.64) to (5.62) we get

$$\begin{aligned} \mathbb{E}_x\{U_p(X_{T(y)}); \tau_{\hat{x}} > T(y), X_{T(y)} \leq y + s(y)\} \\ = (1 + o(1))U_p(y)\mathbb{P}_x\{\tau_{\hat{x}} > T(y)\} + o(1). \end{aligned}$$

Combining this with (5.63), we obtain from (5.61) the following equality, as  $y \rightarrow \infty$ ,

$$\mathbb{E}_x\{U_p(X_{T(y)}); \tau_{\hat{x}} > T(y)\} = (1 + o(1))U_p(y)\mathbb{P}_x\{\tau_{\hat{x}} > T(y)\} + o(1).$$

Plugging this into (5.60) gives

$$U_p(y)\mathbb{P}_x\{\tau_{\hat{x}} > T(y)\} \rightarrow W(x) \quad \text{as } y \rightarrow \infty,$$

which completes the proof due to  $U_p(y) \sim W(y)$ .  $\square$

## 5.4 Pre-stationary distribution of positive recurrent chain with power-like stationary measure

In this section we assume that the distribution of  $X_n$  converges in total variation distance to the unique invariant distribution  $\pi$  as  $n \rightarrow \infty$ , that is,

$$\sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}\{X_n \in A\} - \pi(A)| \rightarrow 0 \quad \text{as } n \rightarrow \infty; \quad (5.65)$$

for a countable Markov chain  $X_n$  this condition holds automatically provided the chain is irreducible, non-periodic, and positive recurrent; for a real-valued chain it is related to Harris ergodicity, see e.g. [56].

**Theorem 5.20.** *Assume that all the conditions of Theorem 5.1 are valid and  $X_n$  is positive recurrent. Then*

$$\mathbb{P}\{X_n > x\} = (F(n/x^2) + o(1))\pi(x, \infty)$$

as  $x \rightarrow \infty$  uniformly for all  $n$ , where

$$F(u) := \frac{I(u) + \rho \int_1^\infty z^{1-\rho} I(u/z^2) dz}{2I(\infty)^{\frac{\rho-1}{\rho-2}}}$$

is a continuous distribution function;  $I(u)$  is defined in Theorem 3.8. In particular, if  $n/x^2 \rightarrow u \in (0, \infty)$  then

$$\mathbb{P}\{X_n > x\} \sim F(u)\pi(x, \infty),$$

and if  $n/x^2 \rightarrow \infty$  then

$$\mathbb{P}\{X_n > x\} \sim \pi(x, \infty).$$

*Proof.* Splitting the path at the last visit to  $B = [0, \hat{x}]$ , we have

$$\begin{aligned}\mathbb{P}\{X_n \in dz\} &= \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} \mathbb{P}_y\{X_j \in dz, \tau_B > j\} \\ &= \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} \frac{U_p(y)}{U_p(z)} Q^j(y, dz).\end{aligned}$$

Consequently,

$$\mathbb{P}\{X_n > x\} = \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} U_p(y) \int_x^\infty \frac{1}{U_p(z)} Q^j(y, dz). \quad (5.66)$$

Fix a sequence  $N_x \rightarrow \infty$  such that  $N_x = o(x^2)$ . Then, since  $Q$  is substochastic and  $U_p$  is increasing,

$$\begin{aligned}\sum_{j=n-N_x+1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} U_p(y) \int_x^\infty \frac{1}{U_p(z)} Q^j(y, dz) &\leq N_x \frac{U_p(\hat{x})}{U_p(x)} \\ &= o(x^2/U_p(x)).\end{aligned} \quad (5.67)$$

Furthermore, for every  $j \leq n - N_x$ , the distribution of  $X_{n-j}$  converges in total variation to  $\pi$ , see (5.65). Therefore,

$$\begin{aligned}\sum_{j=1}^{n-N_x} \int_B \mathbb{P}\{X_{n-j} \in dy\} U_p(y) \int_x^\infty \frac{1}{U_p(z)} Q^j(y, dz) \\ \sim \sum_{j=1}^{n-N_x} \int_B \pi(dy) U_p(y) \int_x^\infty \frac{1}{U_p(z)} Q^j(y, dz).\end{aligned} \quad (5.68)$$

Similar to (5.67),

$$\sum_{j=n-N_x+1}^n \int_B \pi(dy) U_p(y) \int_x^\infty \frac{1}{U_p(z)} Q^j(y, dz) = o(x^2/U_p(x)) \quad (5.69)$$

Combining (5.66)–(5.69), we obtain

$$\begin{aligned}\mathbb{P}\{X_n > x\} &= \int_B \pi(dy) U_p(y) \int_x^\infty \frac{1}{U_p(z)} \sum_{j=1}^n Q^j(y, dz) + o(x^2/U_p(x)) \\ &= c^* \int_x^\infty \frac{\hat{H}_n^{(q)}(dz)}{U_p(z)} + o(x^2/U_p(x)),\end{aligned} \quad (5.70)$$

where

$$\hat{H}_n^{(q)}(A) := \sum_{j=1}^n \mathbb{E}\{e^{-\sum_{i=0}^{j-1} q(\hat{X}_i)}; \hat{X}_j \in A\},$$

where  $\widehat{X}_0$  has distribution (4.32).

Combining Lemma 3.4, Lemma 5.7 and Theorem 3.8, we get

$$\widehat{H}_n^{(q)}(\widehat{x}, x] \sim \mathbb{E}h(\widehat{X}_0)I(n/x^2)x^2 \quad \text{as } x \rightarrow \infty \text{ uniformly for all } n.$$

Integration by parts together with last equivalence implies

$$\begin{aligned} \int_x^\infty \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} &= \frac{\widehat{H}_n^{(q)}(\widehat{x}, x]}{U_p(x)} - \int_x^\infty \widehat{H}_n^{(q)}(\widehat{x}, z] d\frac{1}{U_p(z)} \\ &\sim \mathbb{E}h(\widehat{X}_0) \left[ \frac{I(n/x^2)x^2}{U_p(x)} - \int_x^\infty I(n/z^2)z^2 d\frac{1}{U_p(z)} \right]. \end{aligned}$$

Taking into account that

$$-\frac{d}{dz} \frac{1}{U_p(z)} = \frac{U'_{-1}(z)}{U_p^2(z)} = \frac{e^{R_{-1}(z)}}{U_p^2(z)} \sim \frac{2\mu + b}{bzU_p(z)} \quad \text{as } z \rightarrow \infty$$

owing to (5.14), we deduce

$$\begin{aligned} \int_x^\infty \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} &\sim \mathbb{E}h(\widehat{X}_0) \left[ \frac{I(n/x^2)x^2}{U_p(x)} + \frac{2\mu + b}{b} \int_x^\infty \frac{I(n/z^2)z}{U_p(z)} dz \right] \\ &= \mathbb{E}h(\widehat{X}_0) \left[ \frac{I(n/x^2)x^2}{U_p(x)} + \frac{2\mu + b}{b} x^2 \int_1^\infty \frac{I(n/x^2 z^2)z}{U_p(xz)} dz \right]. \end{aligned}$$

Since the function  $U_p$  is regularly varying at infinity with index  $\rho = 2\mu/b + 1 > 2$ ,  $U_p(xz)/U_p(x) \rightarrow z^\rho$  as  $x \rightarrow \infty$ . Therefore,

$$\int_x^\infty \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} \sim \mathbb{E}h(\widehat{X}_0) \frac{x^2}{U_p(x)} \left[ I(n/x^2) + \rho \int_1^\infty \frac{I(n/x^2 z^2)}{z^{\rho-1}} dz \right]. \quad (5.71)$$

Plugging (5.71) into (5.70), we obtain

$$\mathbb{P}\{X_n > x\} \sim c^* \mathbb{E}h(\widehat{X}_0) \frac{x^2}{U_p(x)} \left[ I(n/x^2) + \rho \int_1^\infty \frac{I(n/x^2 z^2)}{z^{\rho-1}} dz \right]. \quad (5.72)$$

In Theorem 5.1 we have considered the case  $n = \infty$  where right hand side of (5.72) reads as follows:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}\{X_n > x\} &\sim I(\infty) c^* \mathbb{E}h(\widehat{X}_0) \frac{x^2}{U_p(x)} \left[ 1 + \rho \int_1^\infty \frac{dz}{z^{\rho-1}} \right] \\ &= I(\infty) c^* \mathbb{E}h(\widehat{X}_0) \frac{x^2}{U_p(x)} \frac{2\rho - 2}{\rho - 2}, \end{aligned}$$

which concludes the proof.  $\square$

## 5.5 Pre-stationary distribution of null-recurrent chains

We first prove a theorem on the convergence to  $\Gamma$ -distribution, which covers transient and nul-recurrent chains.

**Theorem 5.21.** *Assume that, for some  $b > 0$  and  $\mu > -b/2$ ,*

$$m_1(x) \sim \mu/x \text{ and } m_2(x) \rightarrow b \text{ as } x \rightarrow \infty \quad (5.73)$$

*and that the family  $\{\xi^2(x), x \geq 0\}$  possesses an integrable majorant  $\Xi$ , that is,  $\mathbb{E}\Xi < \infty$  and*

$$\xi^2(x) \leq_{st} \Xi \text{ for all } x. \quad (5.74)$$

*If  $X_n \rightarrow \infty$  in probability as  $n \rightarrow \infty$ , then  $X_n^2/n$  converges weakly to the  $\Gamma$ -distribution with mean  $2\mu + b$  and variance  $(2\mu + b)2b$ .*

The main difference between this result and Theorem 3.7 consists in the fact that here we have conditions on the asymptotic behaviour of *full* moments  $m_1(x)$  and  $m_2(x)$ . Furthermore, as we have remarked after Theorem 3.7, (5.74) yields (3.39). The main reason of introducing these more restrictive assumptions is the fact that the renewal function of any null-recurrent chain is infinite and, therefore, we can not use homogeneous in time truncations as it has been done in the proof of Theorem 3.7. In order to prove Theorem 5.21 we shall introduce truncations of jumps which depend not only on the spacial coordinate but also on the time.

*Proof of Theorem 5.21.* For any  $n \in \mathbb{N}$ , consider a new Markov chain  $Y_k(n)$ ,  $k = 0, 1, 2, \dots$ , with transition probabilities depending on the parameter  $n$ , whose jumps  $\eta(n, x)$  are just truncations of the original jumps  $\xi(x)$  at level  $x \vee \sqrt{n}$  depending on both point  $x$  and time  $n$ , that is,

$$\eta(n, x) = \min\{\xi(x), x \vee \sqrt{n}\}.$$

Given  $Y_0(n) = X_0$ , the probability of discrepancy between the trajectories of  $Y_k(n)$  and  $X_k$  until time  $n$  is at most

$$\begin{aligned} \mathbb{P}\{Y_k(n) \neq X_k \text{ for some } k \leq n\} &\leq \sum_{k=0}^{n-1} \mathbb{P}\{X_{k+1} - X_k \geq \sqrt{n}\} \\ &\leq n\mathbb{P}\{\Xi \geq n\} \\ &\leq \mathbb{E}\{\Xi; \Xi \geq n\} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (5.75)$$

Since  $X_n \rightarrow \infty$  in probability, (5.75) implies that, for every  $c$ ,

$$\inf_{n > n_0, k \in [n_0, n]} \mathbb{P}\{Y_k(n) > c\} \rightarrow 1 \text{ as } n_0 \rightarrow \infty. \quad (5.76)$$

By the choice of the truncation level,

$$\xi(x) \geq \eta(n, x) \geq \xi(x) - \xi(x)\mathbb{I}\{\xi(x) > x\}.$$

Therefore, by the condition (5.74),

$$\mathbb{E}\eta(n, x) = \mathbb{E}\xi(x) + o(1/x) \quad \text{as } x \rightarrow \infty \text{ uniformly in } n \quad (5.77)$$

and

$$\mathbb{E}\eta^2(n, x) = \mathbb{E}\xi^2(x) + o(1) \quad \text{as } x \rightarrow \infty \text{ uniformly in } n, \quad (5.78)$$

hereinafter we write  $f_1(x, n) = o(f_2(x, n))$  as  $x \rightarrow \infty$  uniformly in  $n$  if

$$\sup_n |f_1(x, n)/f_2(x, n)| \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

In addition, the inequality  $\eta(n, x) \leq x \vee \sqrt{n}$  and the condition (5.74) imply that, for every  $j \geq 3$ ,

$$\mathbb{E}\eta^j(n, x) = o(x^{j-2} + n^{(j-2)/2}) \quad \text{as } x \rightarrow \infty \text{ uniformly in } n. \quad (5.79)$$

Let us compute the mean of the increment of  $Y_k^j(n)$ . For  $j = 2$  we have

$$\begin{aligned} \mathbb{E}\{Y_{k+1}^2(n) - Y_k^2(n) | Y_k(n) = x\} &= \mathbb{E}(2x\eta(n, x) + \eta^2(n, x)) \\ &= 2\mu + b + o(1) \end{aligned}$$

as  $x \rightarrow \infty$  uniformly in  $n$ , by (5.77) and (5.78). Applying now (5.76) we get

$$\mathbb{E}(Y_{k+1}^2(n) - Y_k^2(n)) \rightarrow 2\mu + b \quad \text{as } k, n \rightarrow \infty, k \leq n.$$

Hence,

$$\mathbb{E}Y_n^2(n) \sim (2\mu + b)n \quad \text{as } n \rightarrow \infty. \quad (5.80)$$

Let now  $j = 2i$ ,  $i \geq 2$ . We have

$$\begin{aligned} &\mathbb{E}\{Y_{k+1}^{2i}(n) - Y_k^{2i}(n) | Y_k(n) = x\} \\ &= \mathbb{E}\left(2ix^{2i-1}\eta(n, x) + i(2i-1)x^{2i-2}\eta^2(n, x) + \sum_{l=3}^{2i} x^{2i-l}\eta^l(n, x) \binom{2i}{l}\right) \\ &= i[2\mu + (2i-1)b + o(1)]x^{2i-2} + \sum_{l=3}^{2i} x^{2i-l}\mathbb{E}\eta^l(n, x) \binom{2i}{l} \end{aligned} \quad (5.81)$$

as  $x \rightarrow \infty$  uniformly in  $n$ , by (5.77) and (5.78). Owing to (5.79),

$$\begin{aligned} \sum_{l=3}^{2i} x^{2i-l}\mathbb{E}\eta^l(n, x) \binom{2i}{l} &= \sum_{l=3}^{2i} x^{2i-l}o(x^{l-2} + n^{(l-2)/2}) \\ &= o(x^{2i-2}) + \sum_{l=3}^{2i} x^{2i-l}o(n^{(l-2)/2}) \end{aligned}$$

as  $x \rightarrow \infty$  uniformly in  $n$ . Substituting this into (5.81) with  $x = Y_k(n)$  and taking into account (5.76), we deduce that

$$\begin{aligned} \mathbb{E}\{Y_{k+1}^{2i}(n) - Y_k^{2i}(n)\} &= i[2\mu + (2i-1)b + o(1)]\mathbb{E}Y_k^{2i-2}(n) \\ &\quad + \sum_{l=3}^{2i} \mathbb{E}Y_k^{2i-l}(n)o(n^{(l-2)/2}). \end{aligned} \quad (5.82)$$

In particular, for  $j = 2i = 4$  we get

$$\begin{aligned} \mathbb{E}\{Y_{k+1}^4(n) - Y_k^4(n)\} &= 2(2\mu + 3b)\mathbb{E}Y_k^2(n) + \mathbb{E}Y_k(n)o(\sqrt{n}) + o(n) \\ &\sim 2(2\mu + 3b)(2\mu + b)n, \end{aligned}$$

due to (5.80). It implies that

$$\mathbb{E}Y_n^4(n) \sim (2\mu + 3b)(2\mu + b)n^2 \quad \text{as } n \rightarrow \infty.$$

By induction arguments, we deduce from (5.82) that, as  $n \rightarrow \infty$ ,

$$\mathbb{E}Y_n^{2i}(n) \sim n^i \prod_{k=1}^i (2\mu + (2k-1)b),$$

which yields that  $Y_n^2(n)/n$  weakly converges to Gamma distribution with mean  $2\mu + b$  and variance  $2b(2\mu + b)$ . Together with (5.75) this completes the proof.  $\square$

In the critical case  $\mu = b/2$  we have a different type of limiting behaviour which may be described in terms of function

$$G(x) := \int_{\hat{x}}^x \frac{y}{U(y)} dy,$$

which is slowly varying at infinity because  $U$  is regularly varying with index  $\rho = 2\mu/b + 1 = 2$ .

**Theorem 5.22.** *Let  $X_n$  be a Markov chain on a countable set  $\{z_0 < z_1 < z_2 < \dots\}$ . Let conditions of Theorem 5.1 hold with  $\mu = b/2$  and let  $\mathbb{E}U(\xi(z_k))$  be finite for all  $k \geq 0$ . If  $G(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , then  $G(X_n)/G(\sqrt{n})$  converges weakly as  $n \rightarrow \infty$  to the uniform distribution on the interval  $[0, 1]$ .*

*Proof.* According to Corollary 5.10, the assumption  $G(x) \rightarrow \infty$  implies null-recurrence of  $X_n$ . Furthermore, by Corollary 5.16,

$$U(\sqrt{n})\mathbb{P}_x\{\sigma_{z_0} > n\} \rightarrow C(z_0, x) \quad \text{as } n \rightarrow \infty.$$

Let  $T_k$  be intervals between consequent visits of  $X_n$  to the state  $z_0$ . All these random variables are independent. Moreover,  $T_2, T_3, \dots$  are identically distributed and, for every  $k \geq 2$ ,

$$\mathbb{P}\{T_k > n\} \sim \frac{C(z_0)}{U(\sqrt{n})} \quad \text{as } n \rightarrow \infty.$$

Let  $\theta_n$  denote the corresponding renewal process, that is,

$$\theta_n := \max\{k \geq 1 : T_1 + T_2 + \dots + T_k \leq n\}.$$

Let us also introduce the sequence of overshoots:

$$O_n := n - (T_1 + T_2 + \dots + T_{\theta_n}), \quad n \geq 1.$$

It is clear from the definition of  $\theta_n$  that

$$\mathbb{P}\{O_n = j\} = \mathbb{P}\{X_{n-j} = z_0\} \mathbb{P}\{T_2 > j\} \quad \text{for } 0 \leq j \leq n-1$$

and

$$\mathbb{P}\{O_n = n\} = \mathbb{P}\{T_1 > n\}.$$

Then, for every  $z > z_0$  we have

$$\begin{aligned} \mathbb{P}\{X_n > z\} &= \sum_{j=1}^n \mathbb{P}\{X_{n-j} = z_0\} \mathbb{P}_{z_0}\{X_j > z, \sigma_{z_0} > j\} \\ &= \sum_{j=1}^n \mathbb{P}\{O_n = j\} \mathbb{P}\{X_j > z | \sigma_{z_0} > j\}. \end{aligned}$$

According to Theorem 5.17,

$$\mathbb{P}_{z_0}\{X_j > z | \sigma_{z_0} > j\} = e^{-z^2/2bj} + o(1) \quad \text{as } j \rightarrow \infty$$

uniformly for all  $j$ . In addition, for any fixed  $j$ ,

$$\mathbb{P}_{z_0}\{X_j > z | \sigma_{z_0} > j\} \rightarrow 0 \quad \text{as } z \rightarrow \infty.$$

Therefore,

$$\mathbb{P}_{z_0}\{X_j > z | \sigma_{z_0} > j\} = e^{-z^2/2bj} + o(1) \quad \text{as } z \rightarrow \infty$$

uniformly for all  $j$ . Hence,

$$\mathbb{P}\{X_n > z\} = \mathbb{E} \exp \left\{ -\frac{z^2}{2bO_n} \right\} + o(1) \quad \text{as } z \rightarrow \infty,$$

which implies the following relation, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{P}\left\{\frac{G(X_n)}{G(\sqrt{n})} > y\right\} &= \mathbb{P}\{X_n > G^{-1}(yG(\sqrt{n}))\} \\ &= \mathbb{E} \exp \left\{ -\frac{1}{2b} \left( \frac{G^{-1}(yG(\sqrt{n}))}{\sqrt{O_n}} \right)^2 \right\} + o(1). \end{aligned} \quad (5.83)$$

Since  $\mathbb{P}\{T_2 > n\} \sim C(z_0)/U(\sqrt{n})$ , we get, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \int_0^x \mathbb{P}\{T_2 > y\} dy &\sim C(z_0) \int_{\hat{x}^2}^x \frac{1}{U(\sqrt{y})} dy \\ &= 2C(z_0) \int_{\hat{x}}^{\sqrt{x}} \frac{u}{U(u)} du \\ &= 2C(z_0)G(\sqrt{x}). \end{aligned}$$

Then, by Theorem 6 in Erickson [22], for all  $y \in [0, 1]$ ,

$$\mathbb{P}\left\{\frac{G(\sqrt{O_n})}{G(\sqrt{n})} \leq y\right\} \rightarrow y \quad \text{as } n \rightarrow \infty,$$

or in other words

$$\mathbb{P}\{\sqrt{O_n} \leq G^{-1}(yG(\sqrt{n}))\} \rightarrow y \quad \text{as } n \rightarrow \infty. \quad (5.84)$$

Since  $G$  is a slowly varying function, the inverse function satisfies the relation

$$G^{-1}(tu) = o(G^{-1}(u)) \quad \text{as } u \rightarrow \infty,$$

for any fixed  $0 < t < 1$ , so it follows from (5.84) that

$$\frac{\sqrt{O_n}}{G^{-1}(yG(\sqrt{n}))} \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ with probability } y$$

and

$$\frac{\sqrt{O_n}}{G^{-1}(yG(\sqrt{n}))} \rightarrow \infty \quad \text{as } n \rightarrow \infty \text{ with probability } 1 - y.$$

Therefore,

$$\mathbb{E} \exp\left\{-\frac{1}{2b} \left(\frac{G^{-1}(yG(\sqrt{n}))}{\sqrt{O_n}}\right)^2\right\} \rightarrow 1 - y \quad \text{as } n \rightarrow \infty,$$

which completes the proof, due to (5.83). □



## Chapter 6

# Tail analysis for positive recurrent Markov chains with drift going to zero slower than $1/x$

### 6.1 Main results and discussion

In this chapter we consider a Markov chain  $X_n$  possesses a stationary (invariant) probabilistic distribution and denote this distribution by  $\pi$ . We consider the case where  $\pi$  has unbounded support, that is,  $\pi(x, \infty) > 0$  for every  $x$ .

Hereinafter we assume that the first two moments of the jumps satisfy the conditions

$$m_2(x) \rightarrow b > 0 \quad \text{and} \quad m_1(x)x \rightarrow -\infty \quad \text{as } x \rightarrow \infty. \quad (6.1)$$

In this case the tail of  $\pi$  typically decays faster than any power function, it is usually of Weibullian type.

We first show this for chains with jumps  $\pm 1$  and 0. Fix two positive numbers  $a_+ > a_-$ ,  $\beta \in (0, 1)$  and consider a chain  $X_n$  with

$$p_+(x) = \frac{1}{2} \left( 1 - \frac{a_+}{(x+1)^\beta} \right), \quad p_-(x) = \frac{1}{2} \left( 1 + \frac{a_-}{(x+1)^\beta} \right), \quad x \geq 1.$$

Then, according to (5.1),

$$\pi(x) = \pi(0) \exp \left\{ \sum_{k=1}^x \log \left( \frac{p_+(k)}{p_-(k-1)} \right) \right\}.$$

From the definitions of  $p_\pm$  we get

$$\log \left( \frac{p_+(k)}{p_-(k-1)} \right) = \log \left( 1 - \frac{a_+}{(k+1)^\beta} \right) - \log \left( 1 + \frac{a_-}{k^\beta} \right)$$

Set  $d_\beta := \max\{j : j\beta \leq 1\}$ . Then, by Taylor's theorem,

$$\log\left(\frac{p_+(k)}{p_-(k-1)}\right) = -\sum_{j=1}^{d_\beta} \frac{a_+^j - (-a_-)^j}{j} k^{-j\beta} + O(k^{-(d_\beta+1)\beta}).$$

Therefore,

$$\pi(x) \sim C \exp\left\{-\sum_{j=1}^{d_\beta-1} \frac{a_+^j - (-a_-)^j}{j(1-j\beta)} x^{1-j\beta} - \frac{a_+^{d_\beta} - (-a_-)^{d_\beta}}{d_\beta} \sum_{k=1}^x k^{-d_\beta\beta}\right\}. \quad (6.2)$$

If  $d_\beta\beta < 1$  then we get

$$\pi(x) \sim C \exp\left\{-\sum_{j=1}^{d_\beta} \frac{a_+^j - (-a_-)^j}{j(1-j\beta)} x^{1-j\beta}\right\},$$

and if  $d_\beta\beta = 1$  then

$$\pi(x) \sim C x^q \exp\left\{-\sum_{j=1}^{d_\beta-1} \frac{a_+^j - (-a_-)^j}{j(1-j\beta)} x^{1-j\beta}\right\},$$

where  $q = -(a_+^{d_\beta} - (-a_-)^{d_\beta})/d_\beta$ . In this example we have  $m_1(x) = -(a_+ + a_-)/2(x+1)^\beta$  and  $m_2(x) = 1 - (a_+ - a_-)/2(x+1)^\beta$ . Then, according to (2.1), the density of a diffusion with drift  $-(a_+ + a_-)/(x+1)^\beta$  and diffusion coefficient  $1 - (a_+ - a_-)/2(x+1)^\beta$  is asymptotically equivalent to

$$C \exp\left\{-(a_+ + a_-) \sum_{j=1}^{d_\beta-1} \frac{(a_+ - a_-)^{j-1}}{2^{j-1}(1-j\beta)} x^{1-j\beta} - (a_+ + a_-) \frac{(a_+ - a_-)^{d_\beta-1}}{2^{d_\beta-1}} \int_1^x y^{-d_\beta\beta} dy\right\}.$$

Comparing this expression with (6.2), we see that the main term is the same but all correction terms have different coefficients. Since the correction terms play a role in the case  $\beta \leq 1/2$  ( $d_\beta \geq 2$ ), we conclude that the densities are asymptotically equivalent for  $\beta < 1/2$  only. We also see that if  $\beta \leq 1/2$  then it is not sufficient to know the asymptotic behaviour of the first and second moments.

Our first result concerns the case where, roughly speaking,  $m_1(x) = o(1/\sqrt{x})$  as  $x \rightarrow \infty$ . More precisely, we assume that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = -r(x) + o(p(x)) \quad \text{as } x \rightarrow \infty \quad (6.3)$$

where a decreasing differentiable function  $r(x) > 0$  satisfies  $r(x)x \rightarrow \infty$  as  $x \rightarrow \infty$  and

$$r^2(x) = o(p(x)) \quad \text{as } x \rightarrow \infty, \quad (6.4)$$

where  $p(x) \in [0, r(x)]$  is a decreasing differentiable function which is assumed to be integrable,

$$\int_0^\infty p(x)dx < \infty, \quad (6.5)$$

An increasing function  $s(x)$  is assumed to be of order  $o(1/r(x))$ . In view of (6.1), the condition (6.3) is equivalent to

$$m_1^{[s(x)]}(x) + \frac{m_2^{[s(x)]}(x)}{2}r(x) = o(p(x)) \quad \text{as } x \rightarrow \infty. \quad (6.6)$$

We also assume that

$$|p'(x)| \leq |r'(x)|, \quad |r'(x)| = O(r^2(x)) \quad \text{as } x \rightarrow \infty. \quad (6.7)$$

This condition implies, in particular, that

$$c_r := \sup_{x>0} \frac{r(x/2)}{r(x)} < \infty. \quad (6.8)$$

Indeed, for  $\varepsilon > 0$ ,

$$r((1-\varepsilon)x) = r(x) - \varepsilon x r'(\theta) \leq r(x) + c \varepsilon x r(\theta)/\theta,$$

where  $\theta \in [(1-\varepsilon)x, x]$  and  $c := \sup_x x|r'(x)|/r(x) < \infty$ . Hence,

$$r((1-\varepsilon)x) \leq r(x) + c \frac{\varepsilon}{1-\varepsilon} r((1-\varepsilon)x), \quad (6.9)$$

so that for  $\varepsilon := 1/(2c+1)$  we have

$$r((1-\varepsilon)x) \leq r(x) + r((1-\varepsilon)x)/2, \quad r((1-\varepsilon)x) \leq 2r(x),$$

which is equivalent to (6.8). It also follows from (6.9) that

$$r(x) \sim r(y) \quad \text{if } x, y \rightarrow \infty \text{ and } x/y \rightarrow 1. \quad (6.10)$$

All functions like  $x^{-\beta}$ ,  $x^{-\beta} \log^\alpha x$  with  $\beta \in (0, 1)$  satisfy (6.7) with  $p(x) = 0$ , at least eventually in  $x$ .

Define

$$R(x) := \int_0^x r(y)dy. \quad (6.11)$$

Since  $xr(x) \rightarrow \infty$ ,  $R(x) \rightarrow \infty$ . The function  $R(x)$  is concave because  $r(x)$  is decreasing. Taking into account the equalities

$$\begin{aligned} r(x+y) &= r(x) + \int_0^y r'(x+u)du \\ &= r(x) + O(1) \int_0^y \frac{r(x+u)}{x+u} du \\ &= r(x) + O(1)yr(x)/x \end{aligned} \quad (6.12)$$

as  $x \rightarrow \infty$  uniformly for all  $|y| \leq x/2$ , owing to (6.7) and (6.8), we deduce that

$$\begin{aligned} R\left(x + \frac{c}{r(x)}\right) &= R(x) + \int_0^{c/r(x)} r(x+y)dy \\ &= R(x) + c + O(1/xr(x)) \\ &= R(x) + c + o(1) \quad \text{as } x \rightarrow \infty, \end{aligned} \quad (6.13)$$

for any fixed  $c \in \mathbb{R}$ , so that  $1/r(x)$  is a natural step size responsible for constant increase of the function  $R(x)$ .

Consider the following function

$$U(x) := \int_0^x e^{R(y)} dy \quad \text{for } x \geq 0. \quad (6.14)$$

Note that the function  $U$  solves the equation  $U'' - rU' = 0$ . The function  $U(x)$  is convex. Since

$$\frac{U'(x)}{\left(\frac{1}{r(x)}e^{R(x)}\right)'} = \frac{e^{R(x)}}{\left(1 - \frac{r'(x)}{r^2(x)}\right)e^{R(x)}}$$

and  $|r'(x)| = O(r(x)/x) = o(r^2(x))$  by (6.7), L'Hospital's rule yields that

$$U(x) \sim \frac{1}{r(x)}e^{R(x)} \quad \text{as } x \rightarrow \infty. \quad (6.15)$$

**Theorem 6.1.** *Let  $X_n$  be a positive recurrent Markov chain and let  $\pi(\cdot)$  be its invariant probabilistic measure. Let  $\pi$  have unbounded support, that is,  $\pi(x, \infty) > 0$  for all  $x$ . Let (6.1), (6.4)—(6.7) be valid and, for some increasing  $s(x) = o(1/r(x))$  as  $x \rightarrow \infty$ ,*

$$\mathbb{P}\{\xi(x) < -s(x)\} = o(r(x)p(x)), \quad (6.16)$$

$$\mathbb{E}\{U(x + \xi(x)); \xi(x) > s(x)\} = o(p(x)r(x))U(x), \quad (6.17)$$

$$\sup_x \mathbb{E}\{|\xi(x)|^3; |\xi(x)| \leq s(x)\} < \infty. \quad (6.18)$$

Then there exists  $c > 0$  such that, for any  $h > 0$ ,

$$\pi\left(x, x + \frac{h}{r(x)}\right] \sim c \frac{1 - e^{-h}}{r^2(x)U(x)} \quad \text{as } x \rightarrow \infty.$$

In particular, the tail of the invariant measure is of Weibullian type as follows:

$$\pi(x, \infty) \sim \frac{c}{r^2(x)U(x)} \quad \text{as } x \rightarrow \infty.$$

Notice that the condition (6.4) excludes any function  $r(x)$  which decreases like  $1/\sqrt{x}$  or slower. In case where  $|m_1(x)|$  decreases slower than  $1/\sqrt{x}$ , the conclusion of Theorem 6.1 fails, in general. In this case the answer heavily depends on asymptotic properties of higher moments of the chain jumps.

In order to present the tail asymptotics for the invariant measure for general  $m_1(x)$  we need the following conditions.

Fix some  $\gamma \in \{2, 3, 4, \dots\}$  and assume that there exists a decreasing function  $r(x) \in C^\gamma(\mathbb{R}^+)$  such that  $m_1^{[s(x)]}(x) \sim -\frac{b}{2}r(x)$  as  $x \rightarrow \infty$ ,

$$r^\gamma(x) = o(p(x)) \quad \text{as } x \rightarrow \infty \quad (6.19)$$

and

$$m_1^{[s(x)]}(x) + \sum_{j=2}^{\gamma} \frac{m_j^{[s(x)]}(x)}{j!} r^{j-1}(x) = o(p(x)). \quad (6.20)$$

We also assume that the conditions (6.5) and (6.7) of decreasing differentiable  $p$  hold too. We further assume that the function  $r(x)$  is satisfying the condition (6.7) and that, as  $x \rightarrow \infty$ ,

$$r^{(k)}(x) = o(p(x)), \quad p^{(k)}(x) = o(p(x)) \quad \text{for every } 2 \leq k \leq \gamma. \quad (6.21)$$

Define  $R(x)$  as in (6.11) and  $U(x)$  as in (5.7).

**Theorem 6.2.** *Let  $X_n$  be a positive recurrent Markov chain and let  $\pi(\cdot)$  be its invariant probabilistic measure. Let  $\pi$  have unbounded support, that is,  $\pi(x, \infty) > 0$  for all  $x$ . Let (6.1), (6.19)—(6.21) be valid and, for some increasing  $s(x) = o(1/r(x))$  as  $x \rightarrow \infty$ ,*

$$\mathbb{P}\{\xi(x) < -s(x)\} = o(r(x)p(x)), \quad (6.22)$$

$$\mathbb{E}\{U(x + \xi(x)); \xi(x) > s(x)\} = o(p(x)r(x))U(x), \quad (6.23)$$

$$\sup_x \mathbb{E}\{|\xi(x)|^{\gamma+1}; |\xi(x)| \leq s(x)\} < \infty. \quad (6.24)$$

Then there exists  $c > 0$  such that, for any  $h > 0$ ,

$$\pi\left(x, x + \frac{h}{r(x)}\right] \sim c \frac{1 - e^{-h}}{r^2(x)U(x)} \quad \text{as } x \rightarrow \infty.$$

In particular, the tail of the invariant measure is of Weibullian type as follows:

$$\pi(x, \infty) \sim \frac{c}{r^2(x)U(x)} \quad \text{as } x \rightarrow \infty.$$

Notice that the condition (6.23) is fulfilled if, as  $x \rightarrow \infty$ ,

$$\mathbb{E}\{U(\xi(x)); \xi(x) > s(x)\} = o(p(x)) \quad (6.25)$$

$$\mathbb{P}\{\xi(x) > s(x)\} = o(p(x)r(x)). \quad (6.26)$$

Indeed, since the function  $R(x)$  is concave, for  $y > 0$ ,

$$\begin{aligned} U(x+y) &= U(x) + \int_0^y e^{R(x+z)} dz \\ &\leq U(x) + \int_0^y e^{R(x)+R(z)} dz = U(x) + e^{R(x)} U(y). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\{U(x + \xi(x)); \xi(x) > s(x)\} \\ \leq U(x)\mathbb{P}\{\xi(x) > s(x)\} + e^{R(x)}\mathbb{E}\{U(\xi(x)); \xi(x) > s(x)\}, \end{aligned}$$

where the right hand side is of order  $o(p(x)r(x))U(x)$  by (6.25), (6.26), and (6.15); and (6.23) follows.

Let us demonstrate how the function  $r(x)$  may be constructed under some regularity conditions. Assume that  $m_1^{[s(x)]}(x)$  possesses the following decomposition with respect to some nonnegative decreasing function  $t(x) \in C^\gamma(\mathbb{R}^+)$ :

$$m_1^{[s(x)]} = -t(x) + \sum_{j=2}^{\gamma-1} a_{1,j} t^j(x) + O(p(x)), \quad (6.27)$$

and that, for every  $k = 2, 3, \dots, \gamma$ ,

$$m_k^{[s(x)]}(x) = \sum_{j=0}^{\gamma-k} a_{k,j} t^j(x) + O(t^{1-k}(x)p(x)), \quad (6.28)$$

where this function  $t(x)$  satisfies the conditions (6.7) and (6.21). Then there exists—see Lemma 6.8 below—a solution to the equation (6.20) which may be represented as

$$r(x) = t(x) + \sum_{j=2}^{\gamma-1} r_j t^j(x), \quad (6.29)$$

for some reals  $r_2, \dots, r_{\gamma-1}$ . The function  $r(x)$  satisfies the conditions (6.7) and (6.21). In addition, since its derivative,

$$r'(x) = t'(x) + O(t(x)t'(x)) = t'(x) + o(t'(x)),$$

is non-positive ultimately in  $x$ , we may redefine the function  $t(x)$  on a compact set so that the function  $r(x)$  becomes to be decreasing.

Theorems 6.1 and 6.2 give, at first glance, the same answer:

$$\pi\left(x, x + \frac{h}{r(x)}\right] \sim c \frac{1 - e^{-h}}{r^2(x)U(x)}.$$

The difference consists in the choice of the function  $r(x)$ . In Theorem 6.1 this function should satisfy (6.6), and in Theorem 6.2 we use (6.20) instead of (6.6). In order to explain the difference between (6.20) and (6.6) we consider the case when the first moment behaves regularly. We first assume that (6.6) holds with  $r(x) = x^{-\beta}\ell(x)$ . Due to condition (6.4) we may apply Theorem 6.1 for  $\beta > 1/2$  only. In this case

$$R(x) = \int_0^x y^{-\beta}\ell(y) \sim \frac{1}{1-\beta} x^{1-\beta}\ell(x).$$

Recalling that  $U(x) \sim \frac{1}{r(x)}e^{R(x)}$ , we then get

$$\pi(x, \infty) \sim c \frac{x^\beta}{\ell(x)} \exp \left\{ - \int_0^x y^{-\beta} \ell(y) dy \right\} \quad (6.30)$$

and, in particular,

$$\log \pi(x, \infty) \sim - \frac{1}{1-\beta} x^{1-\beta} \ell(x). \quad (6.31)$$

If  $\beta \leq 1/2$  then we have to use (6.20) with  $\gamma = \min\{k \in \mathbb{Z} : k\beta > 1\}$ . This choice of  $\gamma$  follows from (6.19). In order to have a simpler representation of the answer we assume that (6.27) and (6.28) are valid with  $t(x) = x^{-\beta} \ell(x)$ . As it has been mentioned before,

$$r(x) = x^{-\beta} \ell(x) + \sum_{j=2}^{\gamma} r_j x^{-j\beta} \ell^j(x).$$

Consequently,

$$R(x) = \int_0^x y^{-\beta} \ell(y) dy + \sum_{j=2}^{\gamma} r_j \int_0^x y^{-j\beta} \ell^j(y) dy$$

and

$$\pi(x, \infty) \sim c \frac{x^\beta}{\ell(x)} \exp \left\{ - \int_0^x y^{-\beta} \ell(y) dy + \sum_{j=2}^{\gamma} r_j \int_0^x y^{-j\beta} \ell^j(y) dy \right\}. \quad (6.32)$$

Taking logarithm and comparing with (6.31), we see that logarithmic asymptotics are the same for all  $\beta \in (0, 1)$ . But exact asymptotics are different. If, for example,  $\beta \in (1/3, 1/2]$  and  $\ell(x) \equiv 1$  then we get from (6.32)

$$\pi(x, \infty) \sim c \frac{x^\beta}{\ell(x)} \exp \left\{ - \frac{1}{1-\beta} x^{1-\beta} + \frac{r_2}{1-2\beta} x^{1-2\beta} \right\}.$$

For  $\beta > 1/2$  we have only the first summand in the exponent. Finally, in the borderline case  $\beta = 1/2$  we have

$$\pi(x, \infty) \sim c \frac{x^{\beta+r_2}}{\ell(x)} \exp \left\{ - \frac{1}{1-\beta} x^{1-\beta} \right\},$$

which again differs from the case  $\beta > 1/2$ .

## 6.2 An appropriate Lyapunov function and the corresponding change of measure

The Markov chain  $X_n$  is assumed to be positive recurrent with invariant measure  $\pi$ . Let  $B$  be a Borel set in  $\mathbb{R}^+$  with  $\pi(B) > 0$ ; in our proofs we consider an interval  $(-\infty, x_0]$ . Denote, as above,

$$\tau_B := \min\{n \geq 1 : X_n \in B\}.$$

In this section we construct a Lyapunov function which will be used to derive exact asymptotics of Weibullian type.

Consider the function  $r_p(x) := r(x) - p(x)$ . We have  $0 \leq r_p(x) \leq r(x)$ ; this function is decreasing because

$$r'_p(x) = r'(x) - p'(x) < 0,$$

by the condition (6.7). Define

$$\begin{aligned} R_p(x) &:= \int_0^x r_p(y) dy, & 0 \leq R_p(x) \leq R(x), \\ U_p(x) &:= \int_0^x e^{R_p(y)} dy, & 0 \leq U_p(x) \leq U(x). \end{aligned}$$

Since the function  $r_p(x)$  is decreasing, the function  $R_p(x)$  is concave. Since

$$\int_0^\infty r(y) dy = \infty \quad \text{and} \quad C_p := \int_0^\infty p(y) dy < \infty,$$

we have that

$$R_p(x) = R(x) - C_p + o(1) \quad \text{as } x \rightarrow \infty. \quad (6.33)$$

Therefore,

$$U_p(x) \sim e^{-C_p} U(x) \quad \text{as } x \rightarrow \infty. \quad (6.34)$$

Further, since

$$\frac{U'_p(x)}{\left(\frac{1}{r_p(x)} e^{R_p(x)}\right)'} = \frac{e^{R_p(x)}}{\left(1 - \frac{r'_p(x)}{r_p^2(x)}\right) e^{R_p(x)}}$$

and  $|r'_p(x)| \leq |r'(x)| = O(r(x)/x) = o(r^2(x))$  by (6.7), L'Hospital's rule yields that

$$U_p(x) \sim \frac{1}{r_p(x)} e^{R_p(x)} \sim \frac{1}{r(x)} e^{R(x) - C_p} \quad \text{as } x \rightarrow \infty. \quad (6.35)$$

**Lemma 6.3.** *Under the conditions of Theorem 6.2, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}U_p(x + \xi(x)) - U_p(x) = -p(x)r(x)U_p(x) \left( \frac{m_2^{[s(x)]}(x)}{2} + o(1) \right). \quad (6.36)$$

*Proof.* We start with the following decomposition:

$$\begin{aligned} \mathbb{E}U_p(x + \xi(x)) - U_p(x) &= \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) < -s(x)\} \\ &\quad + \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) > s(x)\}. \end{aligned} \quad (6.37)$$



The first term on the right may be bounded as follows:

$$\begin{aligned} |\mathbb{E}\{U_p(x + \xi(x)) - U_p(x); \xi(x) < -s(x)\}| &\leq U_p(x)\mathbb{P}\{\xi(x) < -s(x)\} \\ &= o(p(x)r(x))U_p(x), \end{aligned} \quad (6.38)$$

due to the condition (6.22). To estimate the second term on the right side of (6.37), we make use of Taylor's theorem:

$$\begin{aligned} &\mathbb{E}\{U_p(x + \xi(x)) - U_p(x); |\xi(x)| \leq s(x)\} \\ &= \sum_{k=1}^{\gamma} \frac{U_p^{(k)}(x)}{k!} m_k^{[s(x)]}(x) + \mathbb{E}\left\{ \frac{U_p^{(\gamma+1)}(x + \theta\xi(x))}{(\gamma+1)!} \xi^{\gamma+1}(x); |\xi(x)| \leq s(x) \right\}, \end{aligned} \quad (6.39)$$

where  $0 \leq \theta = \theta(x, \xi(x)) \leq 1$ . By the construction of  $U_p$ ,

$$U_p'(x) = e^{R_p(x)}, \quad U_p''(x) = r_p(x)e^{R_p(x)} = (r(x) - p(x))e^{R_p(x)}, \quad (6.40)$$

and, for  $k = 3, \dots, \gamma + 1$ ,

$$U_p^{(k)}(x) = (e^{R_p(x)})^{(k-1)} = (r_p^{k-1}(x) + o(p(x)))e^{R_p(x)} \quad \text{as } x \rightarrow \infty,$$

where the remainder terms in the parentheses on the right are of order  $o(p(x))$  by the condition (6.21). By the definition of  $r_p(x)$ ,

$$r_p^{k-1}(x) = (r(x) - p(x))^{k-1} = r^{k-1}(x) + o(p(x)),$$

for  $k \geq 3$ , which implies the relation

$$U_p^{(k)}(x) = (r^{k-1}(x) + o(p(x)))e^{R_p(x)} \quad \text{as } x \rightarrow \infty. \quad (6.41)$$

It follows from the equalities (6.40) and (6.41) that

$$\begin{aligned} \sum_{k=1}^{\gamma} \frac{U_p^{(k)}(x)}{k!} m_k^{[s(x)]}(x) &= e^{R_p(x)} \left( \sum_{k=1}^{\gamma} \frac{r^{k-1}(x)}{k!} m_k^{[s(x)]}(x) + o(p(x)) - p(x) \frac{m_2^{[s(x)]}(x)}{2} \right) \\ &= e^{R_p(x)} \left( o(p(x)) - p(x) \frac{m_2^{[s(x)]}(x)}{2} \right), \end{aligned}$$

by the condition (6.20). Hence, the equivalence (6.35) yields

$$\sum_{k=1}^{\gamma} \frac{U_p^{(k)}(x)}{k!} m_k^{[s(x)]}(x) = -r(x)p(x) \frac{m_2^{[s(x)]}(x)}{2} U_p(x) + o(r(x)p(x))U_p(x). \quad (6.42)$$

Owing the condition (6.21) on the derivatives of  $r(x)$  and the condition (6.19),

$$\begin{aligned} U_p^{(\gamma+1)}(x) &= (r^{\gamma}(x) + o(p(x)))e^{R_p(x)} \\ &= o(p(x))e^{R_p(x)} = o(p(x)r(x))U_p(x). \end{aligned}$$

Then, since  $p(x/2) \leq cp(x)$ , the last term in (6.39) possesses the following bound:

$$\begin{aligned} \left| \mathbb{E} \left\{ \frac{U_p^{(\gamma+1)}(x + \theta\xi(x))}{(\gamma+1)!} \xi^{\gamma+1}(x); |\xi(x)| \leq s(x) \right\} \right| \\ \leq o(p(x)r(x))U_p(x) \mathbb{E}\{|\xi(x)|^{\gamma+1}; |\xi(x)| \leq s(x)\} \\ = o(p(x)r(x))U_p(x), \end{aligned}$$

by the condition (6.18). Therefore, it follows from (6.39) and (6.42) that

$$\begin{aligned} \mathbb{E}\{U_p(x + \xi(x)) - U_p(x); |\xi(x)| \leq s(x)\} \\ = -r(x)p(x) \frac{m_2^{[s(x)]}(x)}{2} U_p(x) + o(p(x)r(x))U_p(x). \end{aligned} \quad (6.43)$$

Finally, the last term in (6.37) is of order  $o(p(x)r(x))U_p(x)$  due to the equivalence (6.34) and the condition (6.23). Substituting this together with (6.38) and (6.43) into (6.37), we arrive at lemma conclusion.  $\square$

### 6.3 Proof of Theorem 6.2

**Lemma 6.4.** *Let the conditions of Theorem 6.2 hold. Then there exists an  $\hat{x}$  such that the mean drift of the function  $U_p(x)$  possesses the following lower and upper bounds*

$$-bp(x)r(x)U_p(x) \leq \mathbb{E}U_p(x + \xi(x)) - U_p(x) \leq 0 \quad \text{for all } x > \hat{x}.$$

Now define a new transition kernel via the following change of measure

$$Q(x, dy) := \frac{U_p(y)}{U_p(x)} \mathbb{P}_x\{X_1 \in dy, \tau_B > 1\}, \quad (6.44)$$

where  $B = (-\infty, \hat{x}]$ . It follows from the upper bound in Lemma 6.4 that

$$Q(x, \mathbb{R}^+) = \frac{\mathbb{E}\{U_p(x + \xi(x)), \tau_B > 1\}}{U_p(x)} \leq \frac{\mathbb{E}U_p(x + \xi(x))}{U_p(x)} \leq 1$$

for all  $x > \hat{x}$ . In other words,  $Q$  is a substochastic kernel on  $(\hat{x}, \infty)$ . Furthermore, combining the lower bound in Lemma 6.4 with the estimate

$$\mathbb{E}\{U_p(x + \xi(x)), \tau_B = 1\} \leq U_p(\hat{x}) \mathbb{P}\{x + \xi(x) \leq \hat{x}\} = o(p(x)r(x)),$$

we obtain that

$$q(x) := -\log Q(x, \mathbb{R}^+) = O(p(x)r(x)). \quad (6.45)$$

Let us consider the following normalised kernel

$$\hat{P}(x, dy) = \frac{Q(x, dy)}{Q(x, \mathbb{R}^+)}$$

and let  $\{\widehat{X}_n\}$  denote the corresponding Markov chain; let  $\widehat{\xi}(x)$  be its jump from the state  $x$ . Consequently, performing the inverse change of measure we arrive at the following basic equality:

$$\begin{aligned}\mathbb{P}_x\{X_n \in dy, \tau_B > n\} &= \frac{U_p(x)}{U_p(y)} Q^n(x, dy) \\ &= \frac{U_p(x)}{U_p(y)} \mathbb{E}_x\{e^{-\sum_{k=0}^{n-1} q(\widehat{X}_k)}; \widehat{X}_n \in dy\}.\end{aligned}\quad (6.46)$$

**Lemma 6.5.** *Under the conditions of Theorem 6.2, as  $x \rightarrow \infty$ ,*

$$\mathbb{E}\{\widehat{\xi}(x); |\widehat{\xi}(x)| \leq s(x)\} \sim \frac{b}{2}r(x), \quad (6.47)$$

$$\mathbb{E}\{(\widehat{\xi}(x))^2; |\widehat{\xi}(x)| \leq s(x)\} \rightarrow b, \quad (6.48)$$

$$\mathbb{P}\{|\widehat{\xi}(x)| > s(x)\} = o(p(x)r(x)). \quad (6.49)$$

Moreover, there exists a sufficiently large  $\widehat{x}$  such that

$$\mathbb{E}\{\widehat{\xi}(x); \widehat{\xi}(x) \leq s(x)\} \geq \frac{b}{4}r(x) \quad \text{for all } x \geq \widehat{x}. \quad (6.50)$$

*Proof.* It follows from (6.33) and (6.35) that

$$\frac{U'_p(x)}{U_p(x)} = \frac{e^{R_p(x)}}{U_p(x)} \sim \frac{e^{R(x)-C_p}}{\frac{1}{r(x)}e^{R(x)-C_p}} = r(x)$$

So, the function  $U_p$  satisfies the condition (4.11) with  $c_U = 1$ . Also  $U_p$  satisfies (4.12) for any  $s(x) = o(1/r(x))$  because

$$\frac{U'_p(x+y)}{U'_p(x)} = \frac{e^{R_p(x+y)}}{e^{R_p(x)}} \sim e^{R(x+y)-R(x)} = e^{\int_x^{x+y} r(z)dz} = e^{O(s(x)r(x))} = e^{o(1)}$$

as  $x \rightarrow \infty$  uniformly for all  $|y| \leq s(x)$ , and, by (6.35),

$$\frac{U_p(x+y)}{U_p(x)} \sim \frac{r(x)}{r(x+y)} \frac{e^{R(x+y)}}{e^{R(x)}} \sim e^{R(x+y)-R(x)} \rightarrow 1.$$

Finally,  $U_p$  satisfies (4.13) by Lemma 6.4. So, all conditions of Lemma 4.2 are met and (6.47)–(6.50) follow and the proof is complete.  $\square$

Therefore, the chain  $\widehat{X}_n$  satisfies the conditions (3.69) and (3.70) of the central limit theorem, Theorem 3.14, with  $\widehat{v}(x) = br(x)/2$  and  $\widehat{b} = b$ . Further, the lower bound (6.50) for the drift of  $\widehat{X}_n$  allows to apply Lemma 3.2 to  $\widehat{X}_n$  and to conclude that

$$\mathbb{E}_y \widehat{T}(t) = \mathbb{E}_y \widehat{L}(\widehat{x}, \widehat{T}(t)) < \infty \quad \text{for all } t > y,$$

so hence, for any initial state  $\widehat{X}_0 = y$ ,

$$\mathbb{P}\left\{\limsup_{n \rightarrow \infty} \widehat{X}_n = \infty\right\} = 1.$$

In its turn, then it follows from Theorem 2.18 that  $\widehat{X}_n \rightarrow \infty$  with probability 1.

So, the chain  $\widehat{X}_n$  satisfies all conditions of Theorem 3.14. Furthermore, by Theorem 3.3, there exists a  $c < \infty$  such that

$$\widehat{H}_y(x) := \sum_{n=0}^{\infty} \mathbb{P}_y\{\widehat{X}_n \leq x\} \leq c \frac{1+x}{r(x)} \quad \text{for all } x, y > 0. \quad (6.51)$$

Having this estimate we now prove the following result.

**Lemma 6.6.** *Under the conditions of Theorem 6.2,*

$$h(z) := \lim_{n \rightarrow \infty} \mathbb{E}_z e^{-\sum_{k=0}^n q(\widehat{X}_k)} > 0, \quad z > \widehat{x}.$$

Moreover,  $h(z) \rightarrow 1$  as  $z \rightarrow \infty$ .

*Proof.* The existence of  $h(z)$  is immediate from the monotonicity of the sequence  $e^{-\sum_{k=0}^n q(\widehat{X}_k)}$  in  $n$ . To show positivity it suffices to prove that

$$\mathbb{E}_z \sum_{k=0}^{\infty} q(\widehat{X}_k) < \infty, \quad z > \widehat{x}. \quad (6.52)$$

Note that

$$\mathbb{E}_z \sum_{k=0}^{\infty} q(\widehat{X}_k) = \int_{\widehat{x}}^{\infty} q(y) \widehat{H}_z(dy) \leq c \int_{\widehat{x}}^{\infty} p(y) r(y) \widehat{H}_z(dy),$$

because  $q(y) = O(p(y)r(y))$ . In its turn the latter integral is finite by the following inequality

$$\begin{aligned} \int_{\widehat{x}}^{\infty} p(y) r(y) \widehat{H}_z(dy) &\leq \sum_{j=0}^{\infty} p(2^j \widehat{x}) r(2^j \widehat{x}) \widehat{H}_z(2^{j+1} \widehat{x}) \\ &\leq c \sum_{j=0}^{\infty} p(2^j \widehat{x}) 2^{j+1} \widehat{x}, \end{aligned}$$

due to (6.51). The latter sum may be bounded as

$$\begin{aligned} 4\widehat{x} \sum_{j=0}^{\infty} p(2^j \widehat{x}) 2^{j-1} &\leq 2\widehat{x} p(\widehat{x}) + 4\widehat{x} \sum_{j=1}^{\infty} \sum_{k=2^{j-1}+1}^{2^j} p(k\widehat{x}) \\ &= 2\widehat{x} p(\widehat{x}) + 4\widehat{x} \sum_{k=2}^{\infty} p(k\widehat{x}), \end{aligned}$$

finiteness of the sum on the right follows from the condition (6.5) and (6.52) follows, so that the first statement of the lemma is proven.

To prove the second claim we note that it follows from Theorem 2.15 that, for every fixed  $N > 0$ ,

$$\mathbb{P}_z\{\widehat{X}_n > N \text{ for all } n \geq 1\} \rightarrow 1 \quad \text{as } z \rightarrow \infty,$$

so that

$$\widehat{H}_z(N) \rightarrow 0 \quad \text{as } z \rightarrow \infty.$$

Then, for every fixed  $N$ ,

$$\lim_{z \rightarrow \infty} \mathbb{E}_z \sum_{k=0}^{\infty} q(\widehat{X}_k) \leq \sup_{z > \widehat{x}} \int_N^{\infty} q(y) \widehat{H}_z(dy).$$

As the previous calculations show,

$$\lim_{N \rightarrow \infty} \sup_{z > \widehat{x}} \int_N^{\infty} q(y) \widehat{H}_z(dy) = 0.$$

Therefore, we infer that

$$\lim_{z \rightarrow \infty} \mathbb{E}_z \sum_{k=0}^{\infty} q(\widehat{X}_k) = 0.$$

From this relation and Jensen inequality we finally conclude  $\lim_{z \rightarrow \infty} h(z) = 1$ .  $\square$

Define measure  $H_z^q$  on  $(\widehat{x}, \infty)$  by

$$H_z^q(dy) := \sum_{n=1}^{\infty} \mathbb{E}_z \left\{ e^{-\sum_{k=0}^{n-1} q(\widehat{X}_k)}; \widehat{X}_n \in dy \right\}. \quad (6.53)$$

Combining Lemmas 3.4 and 6.6 and Theorem 3.20 we get the following

**Corollary 6.7.** *For every fixed  $z \geq 0$  and  $h > 0$ ,*

$$H_z^q\left(x, x + \frac{h}{r(x)}\right] \sim h(z) \widehat{H}_z\left(x, x + \frac{h}{r(x)}\right] \quad \text{as } x \rightarrow \infty.$$

We again use the representation (4.33) applied to the test function  $U_p$  which reads

$$\pi(x, x + h/r(x)] = c^* \int_x^{x+h/r(x)} \frac{H^q(dy)}{U_p(y)},$$

where  $H^q$  is defined in (4.31). Notice that, in estimating the last integral, it is useless to follow the integration by parts because of Weibullian nature of the function  $U_p(y)$  while integration only works well for regularly varying functions. By this reason we proceed with dividing the interval  $(x, x + h/r(x))$  into small equal subintervals. So, let us fix large  $m \in \mathbb{Z}^+$  and consider points

$$x_k(m) = x + \frac{k-1}{m} \frac{h}{r(x)}, \quad k \in \{1, 2, \dots, m\}.$$

Then

$$\int_x^{x+h/r(x)} \frac{H^q(dy)}{U_p(y)} = \sum_{k=1}^m \int_{x_k(m)}^{x_{k+1}(m)} \frac{H^q(dy)}{U_p(y)}.$$

Since the function  $U_p(y)$  is increasing, we have the following lower and upper bounds

$$\frac{H^q(x_k(m), x_{k+1}(m))}{U_p(x_{k+1}(m))} \leq \int_{x_k(m)}^{x_{k+1}(m)} \frac{H^q(dy)}{U_p(y)} \leq \frac{H^q(x_k(m), x_{k+1}(m))}{U_p(x_k(m))}.$$

The interval  $[x_k(m), x_{k+1}(m))$  is asymptotically almost the same interval as  $[x_k(m), x_k(m) + \frac{h}{m} \frac{1}{r(x_k(m))})$ . Hence, for every fixed  $m$ , it follows from Theorem 3.4 that, as  $x \rightarrow \infty$ ,

$$\begin{aligned} H^q(x_k(m), x_{k+1}(m)) &\sim \hat{H}(x_k(m), x_{k+1}(m)) \int_B h(z) \mathbb{P}\{\hat{X}_0 \in dz\} \\ &= \hat{H}(x_k(m), x_{k+1}(m)) \frac{\int_B h(z) U_p(z) \pi(dz)}{\int_B U_p(z) \pi(dz)}. \end{aligned}$$

In its turn, Theorem 3.20 yields asymptotics

$$H^q(x_k(m), x_{k+1}(m)) \sim c \frac{h}{mr^2(x_k(m))} \quad \text{as } x \rightarrow \infty,$$

where

$$c := \frac{2 \int_B h(z) U_p(z) \pi(dz)}{b \int_B U_p(z) \pi(dz)}.$$

This implies the following asymptotic upper bound

$$\int_x^{x+h/r(x)} \frac{H^q(dy)}{U_p(y)} \leq (c + o(1)) \frac{h}{m} \sum_{k=1}^{\infty} \frac{1}{r^2(x_k(m)) U_p(x_k(m))}.$$

Substituting the asymptotic relation (6.35) for  $U_p$ , we arrive at the following upper bound:

$$\int_x^{x+h/r(x)} \frac{H^q(dy)}{U_p(y)} \leq (c + o(1)) \frac{h}{m} \sum_{k=1}^m \frac{e^{-R(x_k(m))}}{r(x_k(m))} \quad \text{as } x \rightarrow \infty.$$

Letting  $m \rightarrow \infty$  we approximate the sum on the right multiplied by  $h/m$  by the integral

$$\begin{aligned} r(x) \int_x^{x+h/r(x)} \frac{e^{-R(y)}}{r(y)} dy &\sim \int_0^{h/r(x)} e^{-R(x+y)} dy \\ &= \frac{1}{r(x)} \int_0^h e^{-R(x+y/r(x))} dy \\ &\sim \frac{1}{r(x)} e^{-R(x)} \int_0^h e^{-y} dy = \frac{1 - e^{-h}}{r(x)} e^{-R(x)} \quad \text{as } x \rightarrow \infty, \end{aligned}$$

where we make use of (6.13). In this way the upper bound of Theorem 6.2 is done.

The corresponding lower bound may be derived in the same way and the proof of Theorem 6.2 is complete.

## 6.4 Sufficient condition for existence of $r(x)$

**Lemma 6.8.** *Assume that  $m_1^{[s(x)]}(x)$  possesses the following decomposition with respect to some nonnegative decreasing function  $t(x) \in C^\gamma(\mathbb{R}^+)$ :*

$$m_1^{[s(x)]}(x) = -t(x) + \sum_{j=2}^{\gamma-1} a_{1,j} t^j(x) + O(p(x)), \quad (6.54)$$

and that, for every  $k = 2, 3, \dots, \gamma$ ,

$$m_k^{[s(x)]}(x) = \sum_{j=0}^{\gamma-k} a_{k,j} t^j(x) + O(t^{1-k}(x)p(x)), \quad (6.55)$$

where this function  $t(x)$  satisfies the conditions (6.7) and (6.21). Then there exists a solution to the equation (6.20) which may be represented as

$$r(x) = t(x) + \sum_{j=2}^{\gamma-1} r_j t^j(x), \quad (6.56)$$

for some reals  $r_2, \dots, r_{\gamma-1}$ .

*Proof.* It is sufficient to find  $r(x)$  satisfying the equality

$$m_1^{[s(x)]}(x) + r(x) + \sum_{j=2}^{\gamma} \frac{1}{j!} m_j^{[s(x)]}(x) r^{j-1}(x) = O(p(x)). \quad (6.57)$$

In order to find the coefficients  $r_j$ , let us substitute (6.54), (6.55) and (6.56) into (6.57). Then we arrive at the following equality:

$$\begin{aligned} 0 = & \left( -t(x) + \sum_{j=2}^{\gamma-1} a_{1,j} t^j(x) \right) + \left( t(x) + \sum_{j=2}^{\gamma-1} r_j t^j(x) \right) \\ & + \sum_{j=2}^{\gamma} \frac{1}{j!} \left( \sum_{k=0}^{\gamma-j} a_{j,k} t^k(x) \right) \left( t(x) + \sum_{j=2}^{\gamma-1} r_j t^j(x) \right)^j. \end{aligned}$$

The coefficient for  $t^2$  equals to  $a_{1,2} + r_2 + a_{2,0}/2$ , which implies  $r_2 = -a_{1,2} - a_{2,0}/2$ . The coefficient for  $t^3$  equals to  $a_{1,3} + r_3 + (a_{2,1} + 2a_{2,0}r_1)/2 + a_{3,0}/6$ , which implies

$$r_3 = -a_{1,3} - (a_{2,1} + 2a_{2,0}r_2)/2 - a_{3,0}/6.$$

All further coefficients may be found in recursive way. □

## 6.5 Pre-stationary distribution

In this section we shall always assume that the distribution of  $X_n$  converges to  $\pi$  in total variation distance.

**Theorem 6.9.** *Assume that all the conditions of Theorem 6.2 are valid. If  $r(x)$  is a regularly varying function at infinity with index  $-\beta \in [-1, 0]$  then, for any fixed  $h > 0$ ,*

$$\frac{\mathbb{P}\{X_n \in (x, x + h/r(x)]\}}{\pi(x, x + h/r(x))} = \Phi\left(\frac{n - V(x)}{\sqrt{b \frac{1+\beta}{1+3\beta} \frac{x}{r^3(x)}}}\right) + o(1)$$

as  $x \rightarrow \infty$  uniformly for all  $n$ , where the function  $V(x)$  is given by

$$V(x) = \int_0^x \left( \sum_{k=2}^{\gamma} \frac{m_k^{[s(y)]}(y)}{(k-2)!k} r^{k-1}(y) \right)^{-1} dy.$$

*Proof.* Splitting all trajectories of  $X$  by the last visit to  $B = [0, \hat{x}]$ , we get

$$\begin{aligned} \mathbb{P}\{X_n \in dz\} &= \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} \mathbb{P}_y\{X_j \in dz, \tau_B > j\} \\ &= \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} \frac{U_p(y)}{U_p(z)} Q^j(y, dz). \end{aligned}$$

Consequently,

$$\mathbb{P}\{X_n \in (x, x + h/r(x)]\} = \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} U_p(y) \int_x^{x+h/r(x)} \frac{1}{U_p(z)} Q^j(y, dz). \quad (6.58)$$

Fix a sequence  $N_x \rightarrow \infty$  such that  $N_x = o(1/r^2(x))$ . Then, since  $Q$  is substochastic and  $U_p$  is increasing,

$$\begin{aligned} \sum_{j=n-N_x+1}^n \int_B \mathbb{P}\{X_{n-j} \in dy\} U_p(y) \int_x^{x+h/r(x)} \frac{1}{U_p(z)} Q^j(y, dz) \\ \leq N_x \frac{U_p(\hat{x})}{U_p(x)} = o\left(\frac{1}{r^2(x)U_p(x)}\right). \end{aligned} \quad (6.59)$$

Furthermore, for all  $j \leq n - N_x$ , the distribution of  $X_{n-j}$  converges in total variation to  $\pi$ , see (5.65). Therefore,

$$\begin{aligned} \sum_{j=1}^{n-N_x} \int_B \mathbb{P}\{X_{n-j} \in dy\} U_p(y) \int_x^{x+h/r(x)} \frac{1}{U_p(z)} Q^j(y, dz) \\ = (1 + o(1)) \sum_{j=1}^{n-N_x} \int_B \pi(dy) U_p(y) \int_x^{x+h/r(x)} \frac{1}{U_p(z)} Q^j(y, dz). \end{aligned} \quad (6.60)$$



Similar to (6.59),

$$\sum_{j=n-N_x+1}^n \int_B \pi(dy) U_p(y) \int_x^{x+h/r(x)} \frac{1}{U_p(z)} Q^j(y, dz) = o\left(\frac{1}{r^2(x)U_p(x)}\right). \quad (6.61)$$

Combining (6.58)—(6.61), we obtain

$$\begin{aligned} & \mathbb{P}\{X_n \in (x, x + h/r(x))\} \\ &= (1 + o(1)) \int_B \pi(dy) U_p(y) \int_x^{x+h/r(x)} \frac{1}{U_p(z)} \sum_{j=1}^n Q^j(y, dz) + o\left(\frac{1}{r^2(x)U_p(x)}\right) \\ &= (c^* + o(1)) \int_x^{x+h/r(x)} \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} + o\left(\frac{1}{r^2(x)U_p(x)}\right), \end{aligned} \quad (6.62)$$

where  $\widehat{X}_0$  is defined in (4.32) and

$$\widehat{H}_n^{(q)}(A) := \sum_{j=1}^n \mathbb{E}\{e^{-\sum_{i=0}^{j-1} q(\widehat{X}_i)}; \widehat{X}_j \in A\}.$$

Moreover, the distribution of  $\widehat{X}_0$  and the constant  $c$  are the same as in the proof of Theorem 6.2.

Since  $U_p$  is increasing, we deduce the following lower and upper bounds

$$\frac{H_n^{(q)}(x, x + h/r(x))}{U_p(x + h/r(x))} \leq \int_x^{x+h/r(x)} \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} \leq \frac{H_n^{(q)}(x, x + h/r(x))}{U_p(x)}. \quad (6.63)$$

In order to apply Theorem 3.22 we need to find a regularly varying decreasing function  $v(x)$  such that  $v'(x) = O(v(x)/x)$  and

$$\widehat{m}_1^{[s(x)]}(x) := \mathbb{E}\{\widehat{\xi}(x); |\widehat{\xi}(x)| \leq s(x)\} = v(x) + o(\sqrt{v(x)/x}). \quad (6.64)$$

By definition of  $\widehat{\xi}(x)$ ,

$$\widehat{m}_1^{[s(x)]}(x) = \frac{\mathbb{E}\{U_p(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\}}{Q(x, \mathbb{R}^+)U_p(x)}. \quad (6.65)$$

By Taylor's theorem,

$$\begin{aligned} & \mathbb{E}\{U_p(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\} \\ &= \sum_{k=1}^{\gamma} \frac{U_p^{(k-1)}(x)}{(k-1)!} m_k^{[s(x)]}(x) + \mathbb{E}\left\{\frac{U_p^{(\gamma)}(x + \theta\xi(x))}{\gamma!} \xi^{\gamma+1}(x); |\xi(x)| \leq s(x)\right\}. \end{aligned}$$

It is clear that assumption (6.24) implies boundedness of functions  $m_k^{[s(x)]}(x)$  for all  $k \leq \gamma + 1$ . From this fact and from (6.40) and (6.41) we infer that

$$\begin{aligned} \mathbb{E}\{U_p(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\} &= U_p(x)m_1^{[s(x)]}(x) + U_p'(x) \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{(k-1)!} r^{k-2}(x) \\ &\quad + o(p(x)r(x))U_p(x) + O(r^\gamma(x))U_p(x). \end{aligned}$$

By (6.19),

$$\begin{aligned} \mathbb{E} \{U_p(x + \xi(x))\xi(x); |\xi(x)| \leq s(x)\} \\ = U_p(x)m_1^{[s(x)]}(x) + U'_p(x) \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{(k-1)!} r^{k-2}(x) + O(p(x))U_p(x). \end{aligned}$$

Substituting this relation into (6.65) and using  $Q(x, \mathbb{R}_+) = 1 + O(r(x)p(x))$ , which is immediate from (6.45), we conclude that

$$\widehat{m}_1^{[s(x)]}(x) = m_1^{[s(x)]}(x) + \frac{U'_p(x)}{r_p(x)U_p(x)} \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{(k-1)!} r^{k-1}(x) + O(p(x)) \quad (6.66)$$

Recalling that  $U'_p(x) = e^{R_p(x)}$  and using  $r'_p(x) = O(r_p(x)/x)$  we get

$$(U'_p(x) - r_p(x)U_p(x))' = -r'_p(x)U_p(x) = O\left(\frac{U_p(x)r_p(x)}{x}\right).$$

Since  $r_p(x)U_p(x) \sim e^{R_p(x)}$ , we have

$$|U'_p(x) - r_p(x)U_p(x)| \leq c_1 \int_1^x \frac{e^{R_p(y)}}{y} dy \quad \text{for some } c_1 < \infty.$$

The derivative of  $U_p(x)/x$  is asymptotically equivalent to  $e^{R_p(x)}/x$  because  $r(x)x \rightarrow \infty$ . Therefore, by L'Hopital's rule,

$$|U'_p(x) - r_p(x)U_p(x)| = O(U_p(x)/x),$$

or, in other words,

$$\frac{U'_p(x)}{r_p(x)U_p(x)} = 1 + O(1/xr(x))$$

Plugging this into (6.66), we obtain

$$\widehat{m}_1^{[s(x)]}(x) = m_1^{[s(x)]}(x) + \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{(k-1)!} r^{k-1}(x) + O(1/x).$$

According to (6.20),

$$m_1^{[s(x)]}(x) = - \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{k!} r^{k-1}(x) + o(p(x)).$$

As a result we have the following asymptotic expansion for the expectation of the threshold at levels  $\pm s(x)$  of jumps for the chain  $\widehat{X}_n$

$$\widehat{m}_1^{[s(x)]}(x) = \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{(k-2)!k} r^{k-1}(x) + O(1/x).$$

Now it is clear that (6.64) is valid with

$$v(x) = \sum_{k=2}^{\gamma} \frac{m_k^{[s(x)]}(x)}{(k-2)!k} r^{k-1}(x),$$

because, for some  $c_2 > 0$ ,

$$x\sqrt{v(x)/x} = \sqrt{v(x)x} \geq c_2\sqrt{r(x)x} \rightarrow \infty \quad \text{as } x \rightarrow \infty,$$

and so

$$1/x = o(\sqrt{v(x)/x}) \quad \text{as } x \rightarrow \infty.$$

The function  $v(x)$  is regularly varying at infinity since

$$\frac{v(x)}{r(x)} \sim \frac{m_2^{[s(x)]}(x)}{2} \rightarrow \frac{b}{2}.$$

Combining Theorem 3.22 and Lemma 3.4, we get

$$H_n^{(q)}\left(x, x + \frac{h}{r(x)}\right] \sim \frac{h}{r^2(x)} \Phi\left(\frac{n - V(x)}{\sqrt{b^{\frac{1+\beta}{1+3\beta}} \frac{x}{r^3(x)}}}\right) + o\left(\frac{1}{r^2(x)}\right).$$

From this estimate and  $U_p(x + h/r(x)) \sim e^h U_p(x)$ —as follows from (6.13)—we infer from (6.63) that

$$U_p(x)r^2(x) \int_x^{x+h/r(x)} \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} \leq h\Phi\left(\frac{n - V(x)}{\sqrt{b^{\frac{1+\beta}{1+3\beta}} \frac{x}{r^3(x)}}}\right) + o(1)$$

and

$$U_p(x)r^2(x) \int_x^{x+h/r(x)} \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} \geq he^{-h}\Phi\left(\frac{n - V(x)}{\sqrt{b^{\frac{1+\beta}{1+3\beta}} \frac{x}{r^3(x)}}}\right) + o(1).$$

Splitting the interval  $(x, x + h/r(x)]$  into smaller intervals as it has been done in Theorem 6.2, one can get

$$U_p(x)r^2(x) \int_x^{x+h/r(x)} \frac{\widehat{H}_n^{(q)}(dz)}{U_p(z)} \sim (1 - e^{-h})\Phi\left(\frac{n - V(x)}{\sqrt{b^{\frac{1+\beta}{1+3\beta}} \frac{x}{r^3(x)}}}\right) + o(1).$$

Plugging this into (6.62), we get

$$\mathbb{P}\{X_n \in (x, x + h/r(x)]\} = c^* \frac{1 - e^{-h}}{r^2(x)U_p(x)} \Phi\left(\frac{n - V(x)}{\sqrt{b^{\frac{1+\beta}{1+3\beta}} \frac{x}{r^3(x)}}}\right) + o\left(\frac{1}{r^2(x)U_p(x)}\right).$$

Combining this with Theorem 6.2, we then get

$$\frac{\mathbb{P}\{X_n \in (x, x + h/r(x)]\}}{\pi(x, x + h/r(x))} = \Phi\left(\frac{n - V(x)}{\sqrt{b^{\frac{1+\beta}{1+3\beta}} \frac{x}{r^3(x)}}}\right) + o(1)$$

and the proof is complete.  $\square$



## Chapter 7

# Applications

### 7.1 Random walks conditioned to stay positive

Let  $S_n$  be a random walk with independent and identically distributed increments  $\xi_k$ , that is,  $S_n = \xi_1 + \xi_2 + \dots + \xi_n$ ,  $n \geq 1$ . Let  $\tau_x$  be the first moment when  $S_n$  starting at  $x$  is non-positive:

$$\tau_x := \min\{n \geq 1 : x + S_n \leq 0\}.$$

We shall assume that the random walk  $S_n$  is oscillating. In particular,  $\mathbb{P}\{\tau_x < \infty\} = 1$  for any starting point  $x$ . Let  $\chi^-$  denote the first weak descending ladder height of  $S_n$ , that is,  $\chi^- = -S_{\tau_0}$ . Let  $V(x)$  denote the renewal function corresponding to weak descending ladder epochs of our random walk:

$$V(x) := 1 + \sum_{k=1}^{\infty} \mathbb{P}\{\chi_1^- + \chi_2^- + \dots + \chi_k^- < x\}$$

where  $\chi_k^-$  are independent copies of  $\chi^-$ .

It is well-known—see e.g. Bertoin and Doney [8]—that  $V(x)$  is a harmonic function for  $S_n$  killed at leaving  $(0, \infty)$ . More precisely,

$$V(x) = \mathbb{E}\{V(x + S_1); \tau_x > 1\}, \quad x \geq 0.$$

This implies that Doob's  $h$ -transform

$$P(x, dy) := \frac{V(y)}{V(x)} \mathbb{P}\{x + S_1 \in dy, \tau_x > 1\}$$

defines a stochastic transition kernel on  $\mathbb{R}^+$ . Let  $X_n$  be the corresponding Markov chain. It is usually called *the random walk conditioned to stay positive*. This definition via Doob's  $h$ -transform is equivalent to the construction of the random walk conditioned to stay positive via the weak limit of conditional distributions, see [8].

We now show that if  $\mathbb{E}\xi_1 = 0$  and  $\mathbb{E}\xi_1^2 =: \sigma^2 \in (0, \infty)$ , then  $X_n$  has asymptotically zero drift. Indeed, it follows from the definition of the kernel  $P$  that

$$\begin{aligned} m_1(x) &:= \frac{1}{V(x)} \mathbb{E}\{V(x + \xi_1)\xi_1; \xi_1 > -x\} \\ &= \frac{1}{V(x)} \mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1; \xi_1 > -x\} + \mathbb{E}\{\xi_1; \xi_1 > -x\} \\ &= \frac{1}{V(x)} \mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1; \xi_1 > -x\} + o(1/x), \end{aligned}$$

by the finiteness of  $\mathbb{E}\xi_1^2$ . This assumption also implies that ladder heights have finite expectation, so by the key renewal theorem,

$$V(x + y) - V(x) \rightarrow \frac{y}{\mathbb{E}\chi^-} \quad \text{as } x \rightarrow \infty, \quad (7.1)$$

and hence  $(V(x + \xi_1) - V(x))\xi_1$  converges a.s. to  $\xi_1^2/\mathbb{E}\chi^-$  as  $x \rightarrow \infty$ . By (7.1),  $\sup_x (V(x + 1) - V(x)) =: c < \infty$  which yields  $|V(x + y) - V(x)| \leq c(|y| + 1)$ . This allows to apply the Lebesgue theorem on dominated convergence and to infer that

$$\mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1; \xi_1 > -x\} \rightarrow \frac{\mathbb{E}\xi_1^2}{\mathbb{E}\chi^-} = \frac{\sigma^2}{\mathbb{E}\chi^-}.$$

By the renewal theorem,  $V(x) \sim x/\mathbb{E}\chi^-$  and hence

$$m_1(x) \sim \frac{\sigma^2}{x} \quad \text{as } x \rightarrow \infty. \quad (7.2)$$

For the second moment we have

$$\begin{aligned} m_2(x) &:= \frac{1}{V(x)} \mathbb{E}\{V(x + \xi_1)\xi_1^2; \xi_1 > -x\} \\ &= \frac{1}{V(x)} \mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1^2; \xi_1 > -x\} + \mathbb{E}\{\xi_1^2; \xi_1 > -x\} \\ &= \frac{1}{V(x)} \mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1^2; \xi_1 > -x\} + \sigma^2 + o(1). \end{aligned}$$

Since

$$|V(x + \xi_1) - V(x)|\xi_1^2 \leq c(1 + |\xi_1|)\xi_1^2 \leq c(1 + x)\xi_1^2 \quad \text{for all } |\xi_1| \leq x$$

and

$$\frac{|V(x + \xi_1) - V(x)|}{V(x)}\xi_1^2 \xrightarrow{a.s.} 0 \quad \text{as } x \rightarrow \infty,$$

we get, again by the Lebesgue theorem,

$$\frac{1}{V(x)} \mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1^2; |\xi_1| \leq x\} \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Therefore,

$$m_2(x) = \frac{1}{V(x)} \mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1^2; \xi_1 > x\} + \sigma^2 + o(1).$$

If  $\mathbb{E}\{\xi_1^3; \xi_1 > 0\}$  is finite, then we may apply the Lebesgue theorem to the expectation over the event  $\{\xi_1 > 0\}$  too and get that  $m_2(x) \rightarrow \sigma^2$  as  $x \rightarrow \infty$ . But if  $\mathbb{E}\{\xi_1^3; \xi_1 > 0\} = \infty$  then  $\mathbb{E}\{(V(x + \xi_1) - V(x))\xi_1^2; \xi_1 > x\}$  is infinite for all  $x \geq 0$ . Therefore,  $m_2(x) \equiv \infty$  for all random walks with  $\mathbb{E}\{\xi_1^3; \xi_1 > 0\} = \infty$ .

Clearly, one can directly show that any random walk conditioned to stay positive is transient while the classical Lamperti criterion for transience—where at least the second moment of jumps is assumed to be finite—is applicable to random walks conditioned to stay positive in the only case where  $\mathbb{E}\{\xi_1^3; \xi_1 > 0\}$  is finite.

Moreover, to the best of our knowledge, all known results on the convergence towards  $\Gamma$ -distribution for Markov chains, see Klebaner, Kersting and [17], assume convergence of  $m_2(x)$ . But it is well-known that the finiteness of  $\sigma^2$  is sufficient for the convergence of  $X_n^2/n$  towards  $\Gamma$ -distribution.

It would be natural if general limiting results for Markov chains with asymptotically zero drift include the known results for random walks conditioned to stay positive as important special example of Markov chain with asymptotically zero drift, otherwise it would look quite strange. It was the main motivation for us to state conditions for convergence to  $\Gamma$ -distribution in terms of truncated moments and tail probabilities.

Repeating the arguments used before for the threshold  $x$ , we can conclude that

$$m_1^{[s(x)]}(x) \sim \frac{\sigma^2}{x}, \quad m_2^{[s(x)]}(x) \rightarrow \sigma^2$$

for any  $s(x) \rightarrow \infty$  satisfying  $s(x) = o(x)$ . Thus, in order to apply our criterion for transience, it remains to show that

$$\mathbb{P}\{\xi(x) < -s(x)\} \leq \frac{p(x)}{x}, \quad (7.3)$$

for some decreasing integrable function  $p$ . According to the construction of  $X_n$ , this is equivalent to the upper bound

$$\frac{1}{V(x)} \mathbb{E}\{V(x + \xi_1); \xi_1 < -s(x)\} \leq \frac{p(x)}{x}.$$

The function  $V$  is increasing and  $V(x) \sim C_0 x$ , hence it suffices to show that

$$\mathbb{P}\{\xi_1 < -s(x)\} \leq \frac{p(x)}{x}$$

for some  $s(x) = o(x)$ . Indeed, by the assumption  $\mathbb{E}\xi_1^2 < \infty$ , there exists an increasing unbounded function  $f$  such that  $\mathbb{E}\xi_1^2 f^2(|\xi_1|) < \infty$ . It is equivalent to the integrability of the decreasing function

$$p(x) := \mathbb{E}\{|\xi_1| f(|\xi_1|); |\xi_1| f(|\xi_1|) > x\}.$$

Then

$$\mathbb{E}\{|\xi_1|; |\xi_1|f(|\xi_1|) > x\} \leq p(x),$$

or, equivalently,

$$\mathbb{E}\{|\xi_1|; |\xi_1| > s(x)\} \leq p(x), \quad (7.4)$$

where  $s(x)$  is the inverse to  $xf(x)$ , so  $s(x) \rightarrow \infty$  and  $s(x) = o(x)$ . Hence, by Chebyshev's inequality,

$$\mathbb{P}\{|\xi_1| > s(x)\} \leq \frac{p(x)}{x}. \quad (7.5)$$

As we have seen above, (7.5) implies (7.3). Thus,  $X_n$  is transient.

To apply Theorem 3.7 on convergence to  $\Gamma$ -distribution, we additionally need to check that

$$\mathbb{P}\{\xi(x) > s(x)\} \leq \frac{p(x)}{x},$$

which is equivalent to

$$\frac{1}{V(x)} \mathbb{E}\{V(x + \xi_1); \xi_1 > s(x)\} \leq \frac{p(x)}{x}.$$

Since  $V$  has asymptotically linear growth, we may reduce the previous condition to

$$\mathbb{E}\{x + \xi_1; \xi_1 > s(x)\} \leq p(x),$$

which follows immediately from (7.4) and (7.5). Therefore, by Theorem 3.7,

$$\frac{X_n^2}{n} \Rightarrow \Gamma_{3/2, 2\sigma^2} \quad \text{as } n \rightarrow \infty. \quad (7.6)$$

In addition, the convergence to  $\Gamma$ -distribution is also accompanied by asymptotics for integral renewal function; by Theorem 3.8,

$$H(0, x] \sim \frac{x^2}{\sigma^2} \quad \text{as } x \rightarrow \infty.$$

Random walks conditioned to stay positive are quite special examples of Markov chains with asymptotically zero drift. Their close connection to ordinary random walks allows one to obtain a number of further results. More precisely, by the definition of the transition kernel of  $X_n$ ,

$$\mathbb{P}_z\{X_n \in dx\} = \frac{V(x)}{V(z)} \mathbb{P}\{z + S_n \in dx, \tau_z > n\}. \quad (7.7)$$

This allows one to use the fluctuation theory of random walks in order to derive results for random walks conditioned to stay positive. For example, Caravenna and Chaumont [13] have proved a functional limit theorem for  $X_n$ , Bryn-Jones and Doney [12] proved a local limit theorem for  $X_n$ . Using results of Doney [20] one can also derive asymptotics of local probabilities of small deviations of  $X_n$ . Finally, results by Jones and Doney [21] can be transferred into asymptotics of large deviation probabilities for a random walk conditioned to stay positive.

For a random walk conditioned to stay positive one can prove the following version of the key renewal theorem.



**Proposition 7.1.** *Assume that  $\mathbb{E}\xi_1 = 0$ ,  $\mathbb{E}\xi_1^2 \in (0, \infty)$  and that the distribution of  $\xi_1$  is non-latticed. Then, for every fixed  $\Delta > 0$ ,*

$$h(x) := H(x + \Delta) - H(x) \sim \frac{2\Delta}{\sigma^2}x \quad \text{as } x \rightarrow \infty.$$

*Proof.* Define

$$u(x) := \mathbb{E} \left[ \sum_{n=1}^{\tau_0-1} \mathbb{I}\{S_n \in (x, x + \Delta]\} \right] = \sum_{n=1}^{\infty} \mathbb{P}\{S_n \in (x, x + \Delta], \tau_0 > n\}.$$

Let  $\chi_k^+$  be independent copies of the first strict ascending ladder height  $\chi^+$ . Then, by the classical duality lemma,

$$\sum_{n=1}^{\infty} \mathbb{P}\{S_n \in (x, x + \Delta], \tau_0 > n\} = \sum_{k=1}^{\infty} \mathbb{P}\{\chi_1^+ + \chi_2^+ + \dots + \chi_k^+ \in (x, x + \Delta]\}.$$

Applying strong renewal theorem, we conclude that

$$u(x) \sim \frac{\Delta}{\mathbb{E}\chi^+}, \quad x \rightarrow \infty. \quad (7.8)$$

This gives the asymptotic for  $h$  in case when  $X_0 = 0$ . Indeed, by (7.7) with  $z = 0$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}_0\{X_n \in (x, x + \Delta]\} &= \sum_{n=1}^{\infty} \int_x^{x+\Delta} V(y) \mathbb{P}\{S_n \in dy, \tau_0 > n\} \\ &\sim V(x) \sum_{n=1}^{\infty} \mathbb{P}\{S_n \in (x, x + \Delta], \tau_0 > n\} = V(x)u(x). \end{aligned}$$

Recalling that  $V(x) \sim x/\mathbb{E}\chi^-$  and using (7.8), we obtain

$$\sum_{n=1}^{\infty} \mathbb{P}_0\{X_n \in (x, x + \Delta]\} \sim \frac{\Delta}{\mathbb{E}\chi^- \mathbb{E}\chi^+} x.$$

Now it remains to use the well-known identity

$$\mathbb{E}\chi^- \mathbb{E}\chi^+ = \frac{\sigma^2}{2}.$$

In view of (7.7), for fixed  $z > 0$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}_z\{X_n \in (x, x + \Delta]\} &\sim \frac{V(x)}{V(z)} \sum_{n=1}^{\infty} \mathbb{P}\{z + S_n \in (x, x + \Delta], \tau_z > n\} \\ &= \mathbb{E} \left[ \sum_{n=1}^{\tau_z} \mathbb{I}\{S_n \in (x, x + \Delta]\} \right]. \end{aligned}$$

Splitting the trajectory of  $S_n$  by descending ladder epoch into independent cycles and recalling the definition of  $u(x)$ , we obtain

$$\mathbb{E} \left[ \sum_{n=1}^{\tau_z} \mathbb{I}\{S_n \in (x, x + \Delta]\} \right] = u(x - z) + \mathbb{E} \left[ \sum_{k=1}^{\theta_z - 1} u(x - z + \chi_1^- + \dots + \chi_k^-) \right], \quad (7.9)$$

where

$$\theta_z := \min\{k \geq 1 : \chi_1^- + \dots + \chi_k^- \geq z\}.$$

By (7.8),

$$\mathbb{E} \left[ \sum_{n=1}^{\tau_z} \mathbb{I}\{S_n \in (x, x + \Delta]\} \right] \sim \frac{\Delta}{\mathbb{E}\chi^+} \mathbb{E}[\theta_z]$$

Noting that  $\mathbb{E}[\theta_z] = V(z)$  and recalling  $V(x) \sim x/\mathbb{E}\chi^+$ , we finally get

$$\sum_{n=1}^{\infty} \mathbb{P}_z\{X_n \in (x, x + \Delta]\} \sim \frac{\Delta}{\mathbb{E}\chi^+ \mathbb{E}\chi^-} x = \frac{2\Delta}{\sigma^2} x$$

for every fixed  $z$ .

In order to have the same asymptotics for any initial distribution of the chain it suffices to show that

$$\sup_z \frac{1}{x} \sum_{n=1}^{\infty} \mathbb{P}_z\{X_n \in (x, x + \Delta]\} < \infty \quad (7.10)$$

for all  $x \geq \hat{x}$ . It follows from (7.7) and (7.9) that

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{P}_z\{X_n \in (x, x + \Delta]\} \\ & \leq \frac{V(x + \Delta)}{V(z)} \left( u(x - z) + \mathbb{E} \left[ \sum_{k=1}^{\theta_z - 1} u(x - z + \chi_1^- + \dots + \chi_k^-) \right] \right). \end{aligned}$$

Since every renewal function is subadditive,  $\sup_x u(x) =: u_0 < \infty$ . Consequently,

$$\sum_{n=1}^{\infty} \mathbb{P}_z\{X_n \in (x, x + \Delta]\} \leq \frac{V(x + \Delta)}{V(z)} u_0 \mathbb{E}\theta_z = V(x + \Delta) u_0.$$

(7.10) follows now from the asymptotic linearity of  $V$ . Thus the proof is complete.  $\square$

## 7.2 Reflected random walks with zero drift

Let  $\eta_n$  be a sequence of independent, identically distributed random variables with zero mean and finite variance. The chain defined by

$$X_{n+1} = |X_n + \eta_{n+1}|, \quad n \geq 0. \quad (7.11)$$

is usually called *reflected random walk*. It follows from (7.11) that

$$\begin{aligned}\xi(x) &= (x + \eta)\mathbb{I}\{x + \eta \geq 0\} - (x + \eta)\mathbb{I}\{x + \eta < 0\} - x \\ &= \eta - 2(x + \eta)\mathbb{I}\{x + \eta < 0\} = \eta - 2(x + \eta)^-.\end{aligned}$$

This representation implies that, for any function  $s(x) < x$ ,

$$\begin{aligned}m_1^{[s(x)]}(x) &= \mathbb{E}[\eta; |\eta| \leq s(x)] + \mathbb{E}[\eta + 2(x + \eta)^-; |\eta + 2(x + \eta)^-| \leq s(x), \eta < -x] \\ &= \mathbb{E}[\eta; |\eta| \leq s(x)] - \mathbb{E}[2x + \eta; -2x - s(x) \leq \eta \leq -2x + s(x)].\end{aligned}$$

From this equality and the assumption  $\mathbb{E}\eta = 0$  we infer that

$$|m_1^{[s(x)]}(x)| \leq \mathbb{E}[|\eta|; |\eta| > s(x)] + s(x)\mathbb{P}\{\eta \leq -2x + s(x)\} \leq 2\mathbb{E}[|\eta|; |\eta| > s(x)].$$

The assumption  $\mathbb{E}\eta^2 < \infty$  implies that there exists function  $s(x) = o(x)$  such that  $\mathbb{E}[|\eta|; |\eta| > s(x)]$  is integrable. Consequently,  $|m_1^{[s(x)]}(x)|$  is also integrable. Taking into account that

$$m_2^{[s(x)]}(x) \rightarrow \mathbb{E}\eta^2 \in (0, \infty),$$

we finally obtain

$$\frac{m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = o(p(x))$$

for some decreasing integrable function  $p(x)$  with  $p'(x) = o(x^{-2})$ . Therefore, the reflected random walk  $X_n$  satisfies (5.5) with  $r(x) \equiv 0$ . This implies that  $U(x) = x$  in this case. Furthermore, the validity of (5.8), (5.9) and (5.10) follows easily from the assumption  $\mathbb{E}\eta^2 < \infty$ . Consequently, we may apply Theorems 5.1, 5.11 and 5.17 to the reflected random walk  $X_n$ :

$$\pi(ax, x] \sim c(1 - a)x \quad \text{as } x \rightarrow \infty, \quad (7.12)$$

$$\mathbb{P}_x\{\tau_{\hat{x}} > n\} \sim \frac{V(x)}{\Gamma(3/2)\sqrt{2\mathbb{E}\eta^2}} n^{-1/2} \quad (7.13)$$

and

$$\mathbb{P}\{X_n > u\sqrt{n} | \tau_{\hat{x}} > n\} \sim e^{-u^2/2\mathbb{E}\eta^2} \quad \text{as } n \rightarrow \infty. \quad (7.14)$$

Asymptotics in (7.13) and (7.14) coincide with that for ordinary random walks, only the function  $V(x)$  can be different. This difference comes from the fact that reflection at zero can happen in such a way that the position after the reflection is again bigger than  $\hat{x}$ .

Relation (7.12) implies that  $X_n$  is null recurrent. Recurrence of reflected random walks with finite second moments of increments has been shown by Kemperman [38]. The non-positivity in case of zero mean is immediate from the fact that ordinary driftless random walk is null-recurrent.

### 7.3 Branching processes

We are going to consider branching processes with reproduction laws depending on the number of particles in the population: If there are  $k$  particles then the offspring of every particle is an independent copy of a random variable  $\zeta(k)$ . Furthermore, we assume that there is a time-homogeneous migration of particles. This will be modelled by a sequence  $\eta_n$  of independent and identically distributed random variables. As a consequence we have the following Markov chain:

$$Z_{n+1} := \left( \sum_{i=1}^{Z_n} \zeta_{n+1,i}(Z_n) + \eta_{n+1} \right)^+, \quad n \geq 0, \quad (7.15)$$

where  $\{\zeta_{n,i}(k), n \geq 0, i \geq 1\}$  are independent copies of  $\zeta(k)$ .

There is also a different way of introducing migration of particles:

$$Y_{n+1} := \sum_{i=1}^{(Y_n + \eta_{n+1})^+} \zeta_{n+1,i}(Y_n), \quad n \geq 0. \quad (7.16)$$

The only difference between these two models consists in the order of branching and migration at every time step. In (7.15) one performs first branching and then migration, and in (7.16) these two mechanisms appear in the reversed order.

We shall assume that offspring random variables  $\zeta(k)$  are such that

$$k(\mathbb{E}\zeta(k) - 1) \rightarrow a \in \mathbb{R} \quad (7.17)$$

and

$$\sigma^2(k) := \mathbb{E}(\zeta(k) - \mathbb{E}\zeta(k))^2 \rightarrow \sigma^2 \in (0, \infty) \quad (7.18)$$

as  $k \rightarrow \infty$ . Under these assumptions one can easily determine the asymptotic behaviour of moments of  $Z_n$ :

$$\begin{aligned} \mathbb{E}\{Z_{n+1} - Z_n | Z_n = k\} &\sim a + \mathbb{E}\eta, \\ \mathbb{E}\{(Z_{n+1} - Z_n)^2 | Z_n = k\} &\sim \sigma^2 k. \end{aligned}$$

For the relation in the second line one has to assume that  $\mathbb{E}\eta^2$  is finite.

Linearly growing variances complicate the analysis of  $Z_n$  significantly. In order to get bounded variances we shall consider the chain

$$X_n := \sqrt{Z_n}, \quad n \geq 0.$$

First of all we are going to show that this Markov chain has asymptotically zero drift and bounded second moments. Our proof of this fact is based on the following technical result.

**Lemma 7.2.** *Set  $S_k = \zeta_1^0(k) + \zeta_2^0(k) + \dots + \zeta_k^0(k)$ , where  $\zeta_j^0(k)$  are independent copies of  $\zeta(k) - \mathbb{E}\zeta(k)$ . Assume that (7.17) and (7.18) are valid. If there exists  $\varepsilon(k) \rightarrow 0$  such that*

$$\mathbb{E}\{\zeta^2(k); \zeta(k) > k\varepsilon(k)\} = o(1), \quad k \rightarrow \infty \quad (7.19)$$

then there exists  $\delta(k) \rightarrow 0$  such that

$$\mathbb{E}\{|S_k|; |S_k| > k\delta(k)\} = o(1) \quad (7.20)$$

and

$$\mathbb{E}\{S_k^2; |S_k| > k\delta(k)\} = o(k). \quad (7.21)$$

*Proof.* According to (7.18), the variances of  $\zeta(k)$  are bounded. Then, by the Fuk-Nagaev inequality, see Corollary 1.11 in [57],

$$\mathbb{P}\{|S_k| > x\} \leq C(r) \left(\frac{k}{x^2}\right)^r + k\mathbb{P}\{|\zeta_1^0(k)| > x/r\} \quad (7.22)$$

for each  $r > 1$ . Therefore,

$$\begin{aligned} & \mathbb{E}\{|S_k|^2; |S_k| \geq x\} \\ &= x^2\mathbb{P}\{|S_k| \geq x\} + 2 \int_x^\infty y\mathbb{P}\{|S_k| \geq y\}dy \\ &\leq C(r) \left[ \frac{k^r}{x^{2r-2}} + 2k^r \int_x^\infty y^{1-2r}dy \right] \\ &\quad + k \left[ x^2\mathbb{P}\{|\zeta_1^0(k)| \geq x/r\} + 2 \int_x^\infty y\mathbb{P}\{|\zeta_1^0(k)| \geq y/r\}dy \right] \\ &\leq \frac{r}{r-1}C(r) \frac{k^r}{x^{2r-2}} + kr^2\mathbb{E}\{(\zeta_1^0(k))^2; |\zeta_1^0(k)| \geq x/r\}. \end{aligned}$$

Take  $r = 3$  and  $x = k\delta(k)$  with some  $\delta(k) \geq 3\varepsilon(k)$ . Noting that

$$\{|\zeta_1^0(k)| \geq k\delta(k)/3\} \subset \{\zeta(k) \geq k\delta(k)/3\}$$

for all  $k$  sufficiently large, we obtain

$$\mathbb{E}\{S_k^2; |S_k| \geq k\delta(k)\} \leq Ck \left( \frac{1}{k^2\delta^4(k)} + \mathbb{E}\{\zeta^2(k); \zeta(k) \geq k\delta(k)/3\} \right). \quad (7.23)$$

Choosing

$$\delta(k) = \max \left\{ k^{-1/4}, 3\varepsilon(k), \mathbb{E}\{\zeta^2(k); \zeta(k) \geq k\varepsilon(k)\} \right\}$$

in (7.23), we get

$$\mathbb{E}\{S_k^2; |S_k| \geq k\delta(k)\} = O(k\delta(k)).$$

(7.20) and (7.21) follow easily from this estimate.  $\square$

We are now in position to determine the asymptotic behaviour of the truncated moments of the chain  $X_n$ .

**Proposition 7.3.** *Assume that (7.17), (7.18) and (7.19) are valid. If  $\mathbb{E}\eta$  is finite then there exists a function  $s(x) = o(x)$  such that the truncated moments of the chain  $X_n$  satisfy, as  $x \rightarrow \infty$ ,*

$$m_1^{[s(x)]}(x) \sim \frac{a + \mathbb{E}\eta - \sigma^2/4}{2x}, \quad (7.24)$$

$$m_2^{[s(x)]}(x) \sim \frac{\sigma^2}{4}. \quad (7.25)$$

Furthermore, if the family  $\zeta(k)$  satisfies the Lindeberg condition

$$\mathbb{E}\{\zeta^2(k); \zeta(k) > \varepsilon\sqrt{k}\} = o(1) \quad \text{for every } \varepsilon > 0, \quad (7.26)$$

then, as  $k \rightarrow \infty$ ,  $\xi(\sqrt{k})$  converges in distribution towards the normal law with zero mean and variance  $\sigma^2/4$ , i.e. the chain  $X_n$  is asymptotically homogeneous.

*Proof.* According to the definition of  $X_n$ ,

$$\xi(\sqrt{k}) = \sqrt{(k + a_k + S_k + \eta)^+} - \sqrt{k},$$

where  $a_k := k(\mathbb{E}\zeta(k) - 1)$ . Then, for any  $s(x) = o(x)$  and all  $k$  sufficiently large,

$$\begin{aligned} A_k &:= \{|\xi(\sqrt{k})| \leq s(\sqrt{k})\} \\ &= \{-2\sqrt{k}s(\sqrt{k}) + s^2(\sqrt{k}) - a_k \leq S_k + \eta \leq 2\sqrt{k}s(\sqrt{k}) + s^2(\sqrt{k}) - a_k\} \\ &\supset \{S_k + \eta \leq \sqrt{k}s(\sqrt{k})\}. \end{aligned} \quad (7.27)$$

Therefore,

$$\begin{aligned} \mathbb{E}\{|S_k + \eta|; A_k^c\} &\leq \mathbb{E}\{|S_k + \eta|; |S_k + \eta| > \sqrt{k}s(\sqrt{k})\} \\ &\leq 2\mathbb{E}\left\{|S_k|; |S_k| \geq \frac{1}{2}\sqrt{k}s(\sqrt{k})\right\} + 2\mathbb{E}\left\{|\eta|; |\eta| \geq \frac{1}{2}\sqrt{k}s(\sqrt{k})\right\}. \end{aligned}$$

Choosing  $s(\sqrt{k}) \geq 2\sqrt{k}\delta(k)$  with  $\delta(k)$  from Lemma 7.2 and applying (7.20), we have

$$\mathbb{E}\left\{|S_k|; |S_k| \geq \frac{1}{2}\sqrt{k}s(\sqrt{k})\right\} = o(1).$$

Furthermore, it follows from the assumption  $\mathbb{E}|\eta| < \infty$  that there exists  $s(x) = o(x)$  such that

$$\mathbb{E}\left\{|\eta|; |\eta| \geq \frac{1}{2}\sqrt{k}s(\sqrt{k})\right\} = o(1).$$

As a result,

$$\mathbb{E}\{|S_k + \eta|; A_k^c\} = o(1).$$

This, in its turn, implies that

$$\mathbb{E}\{S_k + \eta; A_k\} = \mathbb{E}\{S_k + \eta\} - \mathbb{E}\{S_k + \eta; A_k^c\} = \mathbb{E}\eta + o(1). \quad (7.28)$$

Using (7.21), we get

$$\frac{1}{k}\mathbb{E}\{S_k^2; A_k\} = \sigma^2(k) - \frac{1}{k}\mathbb{E}\{S_k^2; A_k^c\} = \sigma^2 + o(1). \quad (7.29)$$

Note also that

$$\frac{1}{k}\mathbb{E}\{|\eta S_k|; A_k\} \leq \frac{1}{k}\mathbb{E}\{|\eta|\}\mathbb{E}\{|S_k|\} = O(k^{-1/2}). \quad (7.30)$$

Moreover, noting that  $A_k \subset \{|\eta| \leq 4\sqrt{k}s(\sqrt{k}) + |S_k|\}$  for all  $k$  sufficiently large, we have

$$\mathbb{E}\{\eta^2; A_k\} \leq \mathbb{E}\{\eta^2; |\eta| \leq 4\sqrt{k}s(\sqrt{k}) + |S_k|\}.$$

By the assumption  $\mathbb{E}|\eta| < \infty$ ,

$$\mathbb{E}\{\eta^2; |\eta| \leq x\} \leq x\mathbb{E}|\eta|.$$

Thus,

$$\mathbb{E}\{\eta^2; A_k\} \leq \mathbb{E}|\eta|(4\sqrt{k}s(\sqrt{k}) + \mathbb{E}|S_k|).$$

It follows from the Jensen inequality that  $\mathbb{E}|S_k| \leq \sigma(k)\sqrt{k}$ . Summarising,

$$\frac{1}{k}\mathbb{E}\{\eta^2; A_k\} = o(1). \quad (7.31)$$

Combining (7.29)–(7.31), we obtain

$$\frac{1}{k}\mathbb{E}\{(S_k + \eta)^2; A_k\} = \sigma^2 + o(1). \quad (7.32)$$

Using Taylor's theorem, we get

$$\xi(\sqrt{k}) = \sqrt{(k + a_k + S_k + \eta)^+} - \sqrt{k} = \frac{1}{2} \frac{a_k + S_k + \eta}{\sqrt{k}} - \frac{1 + o(1)}{8} \frac{(a_k + S_k + \eta)^2}{k^{3/2}}$$

uniformly on the set  $A_k$ . Then, using (7.28) and (7.32), and recalling that  $a_k \rightarrow a$ , we obtain

$$\sqrt{k}m_1^{[s(\sqrt{k})]}(\sqrt{k}) = \sqrt{k}\mathbb{E}\{\xi(\sqrt{k}); A_k\} \rightarrow \frac{a + \mathbb{E}\eta - \sigma^2/4}{2}.$$

To determine the asymptotic behaviour of second moments we note that, uniformly on  $A_k$ ,

$$\xi^2(\sqrt{k}) = \left( \frac{a_k + S_k + \eta}{\sqrt{k} + \sqrt{(k + a_k + S_k + \eta)^+}} \right)^2 = \frac{(a_k + S_k + \eta)^2}{4k} (1 + o(1)).$$

Using (7.32) once again we conclude that

$$m_2^{[s(\sqrt{k})]}(\sqrt{k}) = \mathbb{E}\{\xi^2(\sqrt{k}); A_k\} = \frac{\sigma^2}{4} + o(1).$$

It remains to prove that (7.26) implies the weak convergence of  $\xi(\sqrt{k})$ . Since the variances  $\sigma^2(k)$  are bounded,  $\mathbb{P}\{|S_k| > k^{3/4}\} \rightarrow 0$ . Then

$$\mathbb{P}\{\xi(\sqrt{k}) \in A\} = \mathbb{P}\{\sqrt{k + a_k + S_k + \eta} - \sqrt{k} \in A, |S_k| \leq k^{3/4}, |\eta| \leq k^{3/4}\} + o(1)$$

for every Borel set  $A$ . On the set  $\{|S_k| \leq k^{3/4}, |\eta| \leq k^{3/4}\}$  we have

$$\begin{aligned} & \sqrt{k + a_k + S_k + \eta} - \sqrt{k} \\ &= \frac{a_k + S_k + \eta}{\sqrt{k + a_k + S_k + \eta} + \sqrt{k}} \\ &= \frac{S_k}{2\sqrt{k}} \frac{1}{1 + \sqrt{1 + (a_k + S_k + \eta)/k}} + \frac{a_k + \eta}{\sqrt{k + a_k + S_k + \eta} + \sqrt{k}} \\ &= \frac{S_k}{2\sqrt{k}}(1 + o(1)). \end{aligned}$$

Now it remains to note that (7.26) implies that  $S_k/\sigma(k)\sqrt{k}$  converges towards the standard normal law.  $\square$

**Theorem 7.4.** *Assume that the conditions of Proposition 7.3 are valid. If  $\mathbb{P}\{\eta > 0\} > 0$ ,  $a + \mathbb{E}\eta > \sigma^2/2$  and*

$$k\mathbb{P}\{\zeta(k) > k\varepsilon(k)\} \leq q_k \quad (7.33)$$

for some  $\varepsilon(k) \rightarrow 0$  and some decreasing and summable sequence  $q_k$ , then  $Z_n$  is transient and, moreover,  $\frac{Z_n}{n}$  converges weakly to the  $\Gamma$ -distribution with mean  $a + \mathbb{E}\eta$  and variance  $(a + \mathbb{E}\eta)\sigma^2/2$ .

*Proof.* We are going to apply Theorem 2.18 and Theorem 3.7. The assumption  $\mathbb{P}\{\eta > 0\} > 0$  implies that the chain  $X_n$  is irreducible, that is,  $\limsup X_n = \infty$  with probability one. Further, in view of Proposition 7.3, it remains to show that (3.39) is valid. Clearly, this condition is equivalent to integrability of  $x\mathbb{P}\{|\xi(x)| > s(x)\}$ . Substituting  $x = y^{1/2}$  we infer that integrability of  $x\mathbb{P}\{|\xi(x)| > s(x)\}$  is equivalent to integrability of  $\mathbb{P}\{|\xi(\sqrt{y})| > s(\sqrt{y})\}$ . Consequently, it suffices to show that there exists  $s(x) = o(x)$  such that  $\mathbb{P}\{|\xi(\sqrt{k})| > s(\sqrt{k})\}$  is bounded by a decreasing summable sequence.

It follows from (7.27) that

$$\begin{aligned} \mathbb{P}\{|\xi(\sqrt{k})| > s(\sqrt{k})\} &= \mathbb{P}\{A_k^c\} \leq \mathbb{P}\{|S_k + \eta| > \sqrt{k}s(\sqrt{k})\} \\ &\leq \mathbb{P}\{|S_k| > \sqrt{k}s(\sqrt{k})/2\} + \mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})/2\}. \end{aligned}$$

Since  $\mathbb{E}|\eta| < \infty$ , there exists an increasing unbounded function  $f$  such that  $\mathbb{E}|\eta|f(|\eta|)$  is finite. Therefore, there exists  $\delta(k) \downarrow 0$  such that  $k\delta(k)$  is increasing and

$$\sum_{k=1}^{\infty} \mathbb{P}\{|\eta| > k\delta(k)\} < \infty.$$

Therefore, it remains to prove that  $\mathbb{P}\{|S_k| > \sqrt{k}s(\sqrt{k})/2\}$  can be bounded by a decreasing summable sequence. According to (7.22) with  $r = 2$ ,

$$\mathbb{P}\{|S_k| > \sqrt{k}s(\sqrt{k})/2\} \leq \frac{C}{(s(\sqrt{k}))^4} + k\mathbb{P}\{\zeta(k) > \sqrt{k}s(\sqrt{k})/4\}.$$

Taking into account (7.33), we get the desired upper bound. Consequently, the statement of the corollary follows from Theorem 2.18 and Theorem 3.7.  $\square$



Assume that all the conditions of Theorem 7.4 are valid but  $\mathbb{P}\{\eta \leq 0\} = 1$ . Then 0 is the absorbing state. In order to have a irreducible chain we may also add a transition at zero: If  $X_n = 0$  we put  $X_{n+1} = 1$ . Thus, this transformed chain is transient. This implies that  $q := \mathbb{P}\{Z_n \rightarrow \infty\} > 0$ . Moreover, we may apply Theorem 3.7 to this chain. Since this transformed chain visits 0 finite number of times, we conclude that the distribution of  $Z_n/n$  conditioned on  $\{Z_n \rightarrow \infty\}$  converges to the same  $\Gamma$ -distribution:

$$\mathbb{P}\left\{\frac{Z_n}{n} \leq x\right\} \rightarrow (1 - q) + q\Gamma(x). \quad (7.34)$$

For state-dependent processes without migration the weak convergence to the  $\Gamma$ -distribution has been obtained in several papers. Klebaner [41] has shown this convergence for processes satisfying  $\max_{k \geq 1} \mathbb{E}\zeta^m(k) < \infty$  for all  $m \geq 1$ . Höpfner [32] has proved the same result under weaker moment assumptions. He has shown that (7.34) holds for processes satisfying  $\mathbb{E}\zeta(k) = 1 + a/k$ ,  $|\sigma^2(k) - \sigma^2| = O(1)$  and  $\max_{k \geq 1} \mathbb{E}\zeta^2(k) \log(1 + \zeta(k)) < \infty$ . Restrictions in Theorem 7.4 are significantly weaker than in the papers mentioned above.

**Theorem 7.5.** *Assume that (7.17), (7.18) and (7.19) hold. If  $a + \mathbb{E}\eta < \sigma^2/2$  then the chain  $Z_n$  is recurrent.*

*Proof.* We are going to apply Corollary 2.11 to the chain  $\sqrt{Z_n}$ . So, we have to verify the assumptions in Corollary 2.11.

We start with (2.82). Recalling that  $\xi(\sqrt{k}) = \sqrt{(k + S_k + \eta + a_k)^+} - \sqrt{k}$ , we see that  $\xi(\sqrt{k}) \in [0, \sqrt{k}]$  is equivalent to  $S_k + \eta + a_k \in [0, 3k]$ . Furthermore,

$$\xi(\sqrt{k}) \leq \frac{S_k + \eta + a_k}{2\sqrt{k}} \quad \text{on the set } \xi(\sqrt{k}) \geq 0.$$

Therefore,

$$\begin{aligned} & \mathbb{E}\{\xi^3(\sqrt{k}); \xi(\sqrt{k}) \in [0, \sqrt{k}]\} \\ & \leq \frac{\mathbb{E}\{(S_k + \eta + a_k)^3; S_k + \eta + a_k \in [0, 3k]\}}{8k^{3/2}} \\ & \leq \frac{\mathbb{E}\{(S_k + \eta + a_k)^3; S_k + \eta + a_k \in [0, 3k], |\eta + a_k| \leq k\}}{8k^{3/2}} \\ & \quad + \frac{\mathbb{E}\{(S_k + \eta + a_k)^3; S_k + \eta + a_k \in [0, 3k], |\eta + a_k| > k\}}{8k^{3/2}} \\ & \leq \frac{\mathbb{E}\{(S_k + \eta + a_k)^3; |S_k| \leq 4k, |\eta + a_k| \leq k\}}{8k^{3/2}} + 4k^{3/2}\mathbb{P}\{|\eta + a_k| > k\} \\ & \leq \frac{\mathbb{E}\{|S_k|^3; |S_k| \leq 4k\}}{2k^{3/2}} + \frac{\mathbb{E}\{|\eta + a_k|^3; |\eta + a_k| \leq k\}}{2k^{3/2}} \\ & \quad + 4k^{3/2}\mathbb{P}\{|\eta + a_k| > k\}. \end{aligned} \quad (7.35)$$

It is easy to see that

$$\begin{aligned} \frac{\mathbb{E}\{|\eta + a_k|^3; |\eta + a_k| \leq k\}}{2k^{3/2}} &\leq 4 + \frac{\mathbb{E}\{|\eta + a_k|^3; |\eta + a_k| \in (2\sqrt{k}, k]\}}{2k^{3/2}} \\ &\leq 4 + \sqrt{k} \frac{\mathbb{E}\{|\eta + a_k|; |\eta + a_k| \in (2\sqrt{k}, k]\}}{2}. \end{aligned}$$

Thus, applying the Markov inequality to the last summand in (7.35), we get

$$\begin{aligned} \frac{\mathbb{E}\{|\eta + a_k|^3; |\eta + a_k| \leq k\}}{2k^{3/2}} + 4k^{3/2}\mathbb{P}\{|\eta + a_k| > k\} \\ \leq 4 + 4\sqrt{k}\mathbb{E}\{|\eta + a_k|; |\eta + a_k| > 2\sqrt{k}\} = o(\sqrt{k}) \end{aligned} \quad (7.36)$$

due to the assumption  $\mathbb{E}|\eta| < \infty$ . We now turn to the truncated third moment of  $|S_k|$ . It follows from (7.22) with  $r = 2$  that, for each  $t > 0$ ,

$$\begin{aligned} \mathbb{E}\{|S_k|^3; |S_k| \leq tk\} \\ \leq 3 \int_0^{tk} x^2 \mathbb{P}\{|S_k| > x\} dx \leq k^{3/2} + 3 \int_{\sqrt{k}}^{tk} x^2 \mathbb{P}\{|S_k| > x\} dx \\ \leq k^{3/2} + 3C(2) \int_{\sqrt{k}}^{tk} \frac{k^2}{x^2} dx + 3k \int_{\sqrt{k}}^{tk} x^2 \mathbb{P}\{\zeta(k) > x/2\} dx \\ \leq (1 + 3C(2))k^{3/2} + 3k \int_0^{tk} x^2 \mathbb{P}\{\zeta(k) > x/2\} dx \\ = (1 + 3C(2))k^{3/2} + 8k\mathbb{E}\{\zeta^3(k); \zeta(k) \leq tk/2\} + 8k^4\mathbb{P}\{\zeta(k) > tk/2\}. \end{aligned} \quad (7.37)$$

Setting here  $t = 4$ , we get

$$\frac{\mathbb{E}\{|S_k|^3; |S_k| \leq 4k\}}{k^{3/2}} \leq (1 + 3C(2)) + 8 \frac{\mathbb{E}\{\zeta^3(k); \zeta(k) \leq 2k\}}{k^{1/2}} + 8k^{5/2}\mathbb{P}\{\zeta(k) > 2k\}.$$

Let  $\varepsilon(k) \rightarrow 0$  be as in the condition (7.19). Then,

$$\begin{aligned} \frac{\mathbb{E}\{\zeta^3(k); \zeta(k) \leq 2k\}}{k^{1/2}} + k^{5/2}\mathbb{P}\{\zeta(k) > 2k\} \\ \leq \varepsilon(k)\sqrt{k}\sigma^2(k) + 2k^{1/2}\mathbb{E}\{\zeta^2(k); \zeta(k) \in (\varepsilon(k)k, 2k]\} + k^{1/2}\mathbb{E}\{\zeta^2(k); \zeta(k) > 2k\} \\ = \varepsilon(k)\sqrt{k}\sigma^2(k) + 2k^{1/2}\mathbb{E}\{\zeta^2(k); \zeta(k) > \varepsilon(k)k\} \\ = o(\sqrt{k}). \end{aligned}$$

Therefore,

$$\frac{\mathbb{E}\{|S_k|^3; |S_k| \leq 4k\}}{k^{3/2}} = o(\sqrt{k}).$$

Plugging this and (7.36) into (7.35), we see that  $\sqrt{Z_n}$  satisfies (2.82).

In order to show the validity of (2.83) we first note that

$$\xi(\sqrt{k}) \leq \sqrt{S_k + \eta + a_k} \quad \text{on the set } \xi(\sqrt{k}) > 0.$$

Then, by the Markov inequality,

$$\begin{aligned}\mathbb{E}\{\xi^{\varepsilon/2}(\sqrt{k}); \xi(\sqrt{k}) \geq \sqrt{k}\} &\leq \mathbb{E}\{(S_k + \eta + a_k)^{\varepsilon/4}; S_k + \eta + a_k > 3k\} \\ &\leq (\sqrt{k})^{-2+\varepsilon/2} \mathbb{E}\{(S_k + \eta + a_k); S_k + \eta + a_k > 3k\}.\end{aligned}$$

(2.83) follows now from the relation  $\mathbb{E}\{(S_k + \eta + a_k); S_k + \eta + a_k > 3k\} \rightarrow 0$ .

It remains to prove that

$$\frac{2m_1^{[\sqrt{k}]}(\sqrt{k})}{m_2^{[\sqrt{k}]}(\sqrt{k})} \leq \frac{1-\varepsilon}{\sqrt{k}}$$

for all large  $k$ . This inequality is obvious if  $m_1^{[\sqrt{k}]}(\sqrt{k}) \leq 0$ . For  $m_1^{[\sqrt{k}]}(\sqrt{k}) > 0$  we have

$$\begin{aligned}\frac{2m_1^{[\sqrt{k}]}(\sqrt{k})}{m_2^{[\sqrt{k}]}(\sqrt{k})} &\leq \frac{2m_1^{[\sqrt{k}]}(\sqrt{k})}{m_2^{[s(\sqrt{k})]}(\sqrt{k})} \\ &\leq \frac{2m_1^{[s(\sqrt{k})]}(\sqrt{k})}{m_2^{[s(\sqrt{k})]}(\sqrt{k})} + \frac{2\mathbb{E}\{\xi(\sqrt{k}); \xi(\sqrt{k}) \in (s(\sqrt{k}), \sqrt{k}]\}}{m_2^{[s(\sqrt{k})]}(\sqrt{k})}\end{aligned}$$

It follows from Proposition (7.3) that

$$\frac{2m_1^{[s(\sqrt{k})]}(\sqrt{k})}{m_2^{[s(\sqrt{k})]}(\sqrt{k})} \sim \frac{a + \mathbb{E}\eta - \sigma^2/4}{\sigma^2/4} \frac{1}{\sqrt{k}}.$$

By (2.82),

$$\begin{aligned}&\frac{2\mathbb{E}\{\xi(\sqrt{k}); \xi(\sqrt{k}) \in (s(\sqrt{k}), \sqrt{k}]\}}{m_2^{[s(\sqrt{k})]}(\sqrt{k})} \\ &\leq \frac{2}{s^2(\sqrt{k})} \frac{\mathbb{E}\{\xi^3(\sqrt{k}); \xi(\sqrt{k}) \in (0, \sqrt{k}]\}}{m_2^{[s(\sqrt{k})]}(\sqrt{k})} = o(1/\sqrt{k}).\end{aligned}$$

The desired inequality follows now from the assumption  $a + \mathbb{E}\eta < \sigma^2/2$ .  $\square$

**Theorem 7.6.** Assume that (7.17), (7.18) and (7.19) hold. If  $\mathbb{P}\{\eta > 0\} > 0$ ,  $a + \mathbb{E}\eta \in (0, \sigma^2/2)$  and

$$\max_{k \geq 1} k \mathbb{P}\{\zeta(k) \geq y\sqrt{k}\} \leq q(y) \quad (7.38)$$

with some monotone decreasing function  $q(y)$  satisfying

$$\int_0^\infty yq(y)dy < \infty, \quad (7.39)$$

then the chain  $Z_n$  is null-recurrent and, moreover,  $Z_n/n$  converges weakly to the  $\Gamma$ -distribution with mean  $a + \mathbb{E}\eta$  and variance  $(a + \mathbb{E}\eta)\sigma^2/2$ .

*Proof.* We start by showing that the sequence  $\xi(\sqrt{k})$  possesses a square-integrable majorant. It follows from the definition of  $\xi(\sqrt{k})$  that, for every  $y > 0$ ,

$$\begin{aligned}\mathbb{P}\{\xi(\sqrt{k}) \geq y\} &= \mathbb{P}\{\sqrt{k + S_k + \eta + a_k} \geq \sqrt{k} + y\} \\ &= \mathbb{P}\{S_k + \eta + a_k \geq 2\sqrt{k}y + y^2\} \\ &\leq \mathbb{P}\{S_k \geq \sqrt{k}y\} + \mathbb{P}\{\eta \geq y^2\} + \mathbb{I}\{a_k \geq \sqrt{k}y\}.\end{aligned}\quad (7.40)$$

For the left tail we have

$$\begin{aligned}\mathbb{P}\{\xi(\sqrt{k}) \leq -y\} &= \mathbb{P}\{\sqrt{k + S_k + \eta + a_k} \leq \sqrt{k} - y\} \\ &= \mathbb{P}\{S_k + \eta + a_k \leq -2\sqrt{k}y + y^2\} \\ &\leq \mathbb{P}\{S_k \leq -(2\sqrt{k}y - y^2)/3\} + \mathbb{P}\{\eta \leq -(2\sqrt{k}y - y^2)/3\} \\ &\quad + \mathbb{I}\{a_k \leq -(2\sqrt{k}y - y^2)/3\}.\end{aligned}$$

Since  $\xi(\sqrt{k}) \geq -\sqrt{k}$ , we have to consider  $y \leq \sqrt{k}$  only. But for such values of  $y$  we have  $2\sqrt{k}y - y^2 \geq \sqrt{k}y$  and  $2\sqrt{k}y - y^2 \geq y^2$ . Therefore,

$$\mathbb{P}\{\xi(\sqrt{k}) \leq -y\} \leq \mathbb{P}\{S_k \leq -\sqrt{k}y/3\} + \mathbb{P}\{\eta \leq -y^2/3\} + \mathbb{I}\{a_k \leq -y^2/3\}.$$

Combining this estimate with (7.40), we obtain

$$\mathbb{P}\{|\xi(\sqrt{k})| \geq y\} \leq \mathbb{P}\{|S_k| \geq \sqrt{k}y/3\} + \mathbb{P}\{|\eta| \geq y^2/3\} + \mathbb{I}\{|a_k| \geq y^2/3\}.$$

Using (7.22) with  $r = 2$  we get

$$\mathbb{P}\{|\xi(\sqrt{k})| \geq y\} \leq \frac{C(2)}{y^4} + k\mathbb{P}\{|\zeta^0(k)| \geq \sqrt{k}y\} + \mathbb{P}\{|\eta| \geq y^2/3\} + \mathbb{I}\{|a_k| \geq y^2/3\}.$$

Since  $\zeta^0(k) \geq -\mathbb{E}\zeta(k) = -(1 + a_k/k)$  and the sequence  $a_k$  is bounded, there exists  $y_0$  such that

$$\mathbb{P}\{|\xi(\sqrt{k})| \geq y\} \leq \frac{C(2)}{y^4} + k\mathbb{P}\{\zeta(k) \geq \sqrt{k}y\} + \mathbb{P}\{|\eta| \geq y^2/3\}, \quad y \geq y_0.$$

Using (7.38), we finally get

$$\mathbb{P}\{|\xi(\sqrt{k})| \geq y\} \leq \frac{C(2)}{y^4} + q(y) + \mathbb{P}\{|\eta| \geq y^2/3\}, \quad y \geq y_0.$$

Let  $\Xi$  be a random variable with values in  $[y_0, \infty)$  such that

$$\mathbb{P}\{\Xi \geq y\} = \min \left\{ 1, \frac{C(2)}{y^4} + q(y) + \mathbb{P}\{|\eta| \geq y^2/3\} \right\}, \quad y \geq y_0.$$

Clearly,  $\Xi$  is a stochastic majorant for the sequence  $\xi(\sqrt{k})$ . The finiteness of  $\mathbb{E}\Xi^2$  follows from (7.39) and the assumption  $\mathbb{E}|\eta| < \infty$ .

Having this majorant we may apply Corollary 2.13. Thus,  $X_n$  is null recurrent. This, in its turn, implies that  $X_n \rightarrow \infty$  in probability.

We now determine the behaviour of full moments  $m_1(\sqrt{k})$  and  $m_2(\sqrt{k})$ . We know from Proposition 7.3 that

$$m_1^{[s(\sqrt{k})]}(\sqrt{k}) \sim \frac{a + \mathbb{E}\eta - \sigma^2/4}{2\sqrt{k}}$$

for any  $s(x)$  such that  $s(x)/x \rightarrow 0$  sufficiently slow. From the existence of the majorant we infer that

$$\mathbb{E}\{|\xi(\sqrt{k})|; |\xi(\sqrt{k})| \geq s(\sqrt{k})\} \leq \frac{1}{s(\sqrt{k})} \mathbb{E}\{\Xi^2; \Xi \geq s(\sqrt{k})\}.$$

Consequently, we can choose  $s(x)$  such that  $\mathbb{E}\{|\xi(\sqrt{k})|; |\xi(\sqrt{k})| \geq s(\sqrt{k})\} = o(1/\sqrt{k})$ . This yields

$$m_1(\sqrt{k}) \sim \frac{a + \mathbb{E}\eta - \sigma^2/4}{2\sqrt{k}}.$$

The same argument gives also  $m_2(\sqrt{k}) \sim \sigma^2/4$ . Thus, the convergence of  $Z_n/n$  towards the  $\Gamma$ -distribution follows now from Theorem 5.21.  $\square$

Convergence of critical branching processes with immigration towards the gamma distribution has been first proven by Seneta [64]. More precisely, he has shown that if  $\zeta(k)$  are identically distributed with mean 1 and variance  $\sigma^2$  and if  $\eta$  is non-negative with finite mean then  $Z_n/n$  converges weakly to the  $\Gamma$ -distribution. If  $\mathbb{E}\eta > \sigma^2/2$  then this is a particular case of our Theorem 7.4. If  $\mathbb{E}\eta \leq \sigma^2/2$  then, in order to apply Theorem 7.6, we have to check the validity of (7.38). For identically distributed variables this condition reduces to

$$k\mathbb{P}\{\zeta \geq y\sqrt{k}\} \leq q(y).$$

This estimate follows quite easily from the finiteness of  $\mathbb{E}\zeta^2 \log(1 + \zeta)$ . Indeed,

$$k\mathbb{P}\{\zeta \geq y\sqrt{k}\} \leq \frac{4}{y^2} \int_{y\sqrt{k}/2}^{y\sqrt{k}} u\mathbb{P}\{\zeta > u\}du \leq \frac{4}{y^2} \int_{y/2}^{\infty} u\mathbb{P}\{\zeta > u\}du$$

and it remains to note that  $\frac{1}{y} \int_{y/2}^{\infty} u\mathbb{P}\{\zeta > u\}du$  is integrable if and only if the expectation  $\mathbb{E}\zeta^2 \log(1 + \zeta)$  is finite.

Studying critical and near-critical recurrent branching processes one is usually interested in the asymptotic behaviour of the non-extinction probability and in the limiting behaviour of the process conditioned on the non-extinction. And we have corresponding results for a more general class of Markov chains, see Theorem 5.11 and Corollary 5.16. In order to apply these theorems, we have to find restrictions on  $\zeta(k)$  and  $\eta$  which guarantee (5.2)–(5.4) and (5.8)–(5.10).

The hardest task, from the technical point of view, consists in finding a regular function  $r(x)$  such that (5.3) takes place. In what follows we concentrate on the case when one can take  $r(x) = c/x$ .

We first prove a refined version of Lemma 7.2.

**Lemma 7.7.** *Assume that (7.17) and (7.18) are valid. Assume also that there exists  $\varepsilon(k) \rightarrow 0$  such that*

$$\mathbb{E}\{\zeta(k); \zeta(k) > k\varepsilon(k)\} + \frac{1}{k}\mathbb{E}\{\zeta^2(k); \zeta(k) > k\varepsilon(k)\} \leq q_k \quad (7.41)$$

*for some decreasing summable sequence  $q_k$ . Then there exist  $\delta(k) \rightarrow 0$  and a decreasing summable sequence  $Q_k$  such that*

$$\frac{1}{k}\mathbb{E}\{|S_k|; |S_k| > k\delta(k)\} + \frac{1}{k^2}\mathbb{E}\{S_k^2; |S_k| > k\delta(k)\} \leq Q_k. \quad (7.42)$$

*Proof.* It follows from (7.23) that

$$\frac{1}{k^2}\mathbb{E}\{S_k^2; |S_k| > k\delta(k)\} \leq C \left( \frac{1}{k^3\delta^4(k)} + \frac{1}{k}\mathbb{E}\{\zeta^2(k); \zeta(k) > k\delta(k)/3\} \right).$$

By the same arguments,

$$\frac{1}{k}\mathbb{E}\{|S_k|; |S_k| > k\delta(k)\} \leq C \left( \frac{1}{k^3\delta^5(k)} + \mathbb{E}\{\zeta(k); \zeta(k) > k\delta(k)/3\} \right).$$

Choosing  $\delta(k) = \max\{3\varepsilon(k), k^{-1/5}\}$  and taking into account (7.41) we obtain

$$\frac{1}{k}\mathbb{E}\{|S_k|; |S_k| > k\delta(k)\} + \frac{1}{k^2}\mathbb{E}\{S_k^2; |S_k| > k\delta(k)\} \leq C(k^{-2} + \delta_k).$$

Thus, the proof is complete.  $\square$

**Proposition 7.8.** *Assume that (7.41) is valid. If  $\mathbb{E}|\eta| \log(1+|\eta|)$  is finite and there exists a decreasing summable sequence  $v_k$  such that*

$$\frac{1}{k}|k(\mathbb{E}\zeta(k) - 1) - a| \leq v_k, \quad (7.43)$$

$$\frac{1}{k}|\sigma^2(k) - \sigma^2| \leq v_k, \quad (7.44)$$

$$\frac{1}{k^2}\mathbb{E}\{\zeta^3(k); \zeta(k) \leq k\} \leq v_k, \quad (7.45)$$

*then there exist  $s(x) = o(x)$  and a decreasing integrable function  $p(x)$  such that, as  $x \rightarrow \infty$ ,*

$$m_1^{[s(x)]}(x) = \frac{a + \mathbb{E}\eta - \sigma^2/4}{2x} + o(p(x)), \quad (7.46)$$

$$m_2^{[s(x)]}(x) = \frac{\sigma^2}{4} + o(xp(x)). \quad (7.47)$$

*Proof.* By Taylor's theorem,

$$\xi(\sqrt{k}) = \frac{1}{2} \frac{(a_k + S_k + \eta)}{\sqrt{k}} - \frac{1}{8} \frac{(a_k + S_k + \eta)^2}{k^{3/2}} + \theta \frac{(a_k + S_k + \eta)^3}{k^{5/2}}, \quad (7.48)$$

where  $\theta = \theta(a_k, S_k, \eta) \in (-1, 1)$  on the set  $A_k$ .

We consider the expectation of every summand separately. It is clear that

$$\mathbb{E}\{a_k + S_k + \eta; A_k\} = a_k + \mathbb{E}\eta + \mathbb{E}\{a_k + S_k + \eta; A_k^c\}. \quad (7.49)$$

Recalling that  $A_k^c \subset \{|S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\}$  for all  $k$  sufficiently large and using the Markov inequality, we obtain

$$\begin{aligned} |\mathbb{E}\{a_k + S_k + \eta; A_k^c\}| &\leq |a_k| \mathbb{P}\{A_k^c\} + \mathbb{E}\{|S_k + \eta|; A_k^c\} \\ &\leq |a_k| \mathbb{P}\{|S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\} + \mathbb{E}\{|S_k + \eta|; |S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\} \\ &\leq \left( \frac{|a_k|}{\sqrt{k}s(\sqrt{k})} + 1 \right) \mathbb{E}\{|S_k + \eta|; |S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\} \\ &\leq C \mathbb{E}\{|S_k + \eta|; |S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\}. \end{aligned}$$

Combining this bound with (7.49), we get

$$\mathbb{E} \left\{ \frac{a_k + S_k + \eta}{\sqrt{k}}; A_k \right\} = \frac{a + \mathbb{E}\eta}{\sqrt{k}} + O(g_1(k)), \quad (7.50)$$

where

$$g_1(k) = \frac{1}{\sqrt{k}} |a_k - a| + \frac{1}{\sqrt{k}} \mathbb{E}\{|S_k + \eta|; |S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\}.$$

By Lemma 7.7,

$$\frac{1}{k} \mathbb{E} \left\{ |S_k|; |S_k| > \sqrt{k}s(\sqrt{k})/2 \right\} \leq Q_k$$

for any function  $s(x)$  such that  $s(\sqrt{k}) \geq \sqrt{k}\delta(\sqrt{k})$ . Furthermore, the assumption  $\mathbb{E}|\eta| \log |\eta| < \infty$  implies that the sequence  $\frac{1}{k} \mathbb{E}\{|\eta|; |\eta| > \sqrt{k}s(\sqrt{k})/2\}$  is decreasing and summable provided that  $s(x)/x \rightarrow 0$  sufficiently slow. As a result, the sequence  $\frac{1}{k} \mathbb{E}\{|S_k + \eta|; |S_k + \eta| \geq \sqrt{k}s(\sqrt{k})\}$  possesses a decreasing summable majorant. Combining this with (7.43), we conclude that there exists a decreasing summable sequence  $r_1(k)$  such that

$$\frac{1}{\sqrt{k}} g_1(k) \leq r_1(k) \quad (7.51)$$

for an appropriate choice of  $s(x)$ .

For the second summand in (7.48) we have

$$\begin{aligned} &\mathbb{E} \left\{ \frac{(a_k + S_k + \eta)^2}{k^{3/2}}; A_k \right\} \\ &= \frac{\mathbb{E}\{S_k^2; A_k\}}{k^{3/2}} + \frac{\mathbb{E}\{2S_k\eta; A_k\}}{k^{3/2}} + \frac{\mathbb{E}\{\eta^2; A_k\}}{k^{3/2}} + \frac{2a_k \mathbb{E}\{S_k + \eta; A_k\} + a_k^2}{k^{3/2}}. \end{aligned} \quad (7.52)$$

It is easy to see that

$$\frac{|2a_k \mathbb{E}\{S_k + \eta; A_k\} + a_k^2|}{k^{3/2}} \leq \frac{a_k^2 + 2|a_k|(\mathbb{E}|\eta| + \mathbb{E}|S_k|)}{k^{3/2}} = O\left(\frac{1}{k}\right). \quad (7.53)$$

Using the fact that  $A_k^c \subset \{|S_k + \eta| > \sqrt{k}s(\sqrt{k})\}$  for all large  $k$ , we obtain

$$\begin{aligned} \frac{\mathbb{E}\{S_k^2; A_k\}}{k^{3/2}} &= \frac{\sigma^2(k)}{\sqrt{k}} - \frac{\mathbb{E}\{S_k^2; A_k^c\}}{k^{3/2}} \\ &= \frac{\sigma^2}{\sqrt{k}} + O\left(\frac{|\sigma^2(k) - \sigma^2|}{\sqrt{k}} + \frac{\mathbb{E}\{S_k^2; |S_k + \eta| > \sqrt{k}s(\sqrt{k})\}}{k^{3/2}}\right). \end{aligned} \quad (7.54)$$

Next we note that

$$\frac{|\mathbb{E}\{S_k \eta; A_k\}|}{k^{3/2}} \leq \frac{\mathbb{E}\{|S_k| |\eta|\}}{k^{3/2}} = \frac{\mathbb{E}\{S_k\} \mathbb{E}\{|\eta|\}}{k^{3/2}} = O\left(\frac{1}{k}\right). \quad (7.55)$$

Finally, recalling that  $A_k \subset \{|\eta| \leq 4\sqrt{k}s(\sqrt{k}) + |S_k|\}$ , we have

$$\begin{aligned} \frac{\mathbb{E}\{\eta^2; A_k\}}{k^{3/2}} &\leq \frac{1}{k^{3/2}} \mathbb{E}\{\eta^2; |\eta| \leq 8\sqrt{k}s(\sqrt{k})\} + \frac{1}{k^{3/2}} \mathbb{E}\{\eta^2; |\eta| \leq 2|S_k|, |S_k| > 4\sqrt{k}s(\sqrt{k})\} \\ &\leq \frac{1}{k^{3/2}} \mathbb{E}\{\eta^2; |\eta| \leq 8\sqrt{k}s(\sqrt{k})\} + \frac{4}{k^{3/2}} \mathbb{E}\{S_k^2; |S_k| > 4\sqrt{k}s(\sqrt{k})\}. \end{aligned} \quad (7.56)$$

Combining (7.52)–(7.56), we obtain

$$\frac{\mathbb{E}\{(a_k + S_k + \eta)^2; A_k\}}{k^{3/2}} = \frac{\sigma^2}{\sqrt{k}} + O(g_2(k)), \quad (7.57)$$

where

$$\begin{aligned} g_2(k) &= \frac{1}{k^{3/2}} + \frac{|\sigma^2(k) - \sigma^2|}{k^{1/2}} + \frac{1}{k^{3/2}} \mathbb{E}\{\eta^2; |\eta| \leq 8\sqrt{k}s(\sqrt{k})\} \\ &\quad + \frac{\mathbb{E}\{S_k^2; |S_k + \eta| > \sqrt{k}s(\sqrt{k})\}}{k^{3/2}} + \frac{\mathbb{E}\{S_k^2; |S_k| > \sqrt{k}s(\sqrt{k})/2\}}{k^{3/2}}. \end{aligned}$$

The assumption  $\mathbb{E}|\eta| < \infty$  implies easily that the sequence  $\frac{1}{k^2} \mathbb{E}\{\eta^2; |\eta| \leq 8\sqrt{k}s(\sqrt{k})\}$  is decreasing and summable provided that  $s(x)/x$  converges to zero sufficiently slow.

Obviously,

$$\begin{aligned} &\mathbb{E}\{S_k^2; |S_k + \eta| > \sqrt{k}s(\sqrt{k})\} + \mathbb{E}\{S_k^2; |S_k| > \sqrt{k}s(\sqrt{k})/2\} \\ &\leq 2\mathbb{E}\{S_k^2; |S_k| > \sqrt{k}s(\sqrt{k})/2\} + \mathbb{E}\{S_k^2 \mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})/2\}\} \\ &\leq 2\mathbb{E}\{S_k^2; |S_k| > \sqrt{k}s(\sqrt{k})/2\} + Ck\mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})/2\}. \end{aligned}$$

According to Lemma 7.7, for every  $s(x)$  satisfying  $s(\sqrt{k}) > 2\sqrt{k}\delta(\sqrt{k})$ ,

$$\frac{1}{k^2} \mathbb{E}\{S_k^2; |S_k| > \sqrt{k}s(\sqrt{k})/2\} \leq Q_k.$$

Furthermore, the assumption  $\mathbb{E}|\eta| < \infty$  implies that if  $s(x)/x$  converges to zero sufficiently slow then  $\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})/2\}$  is decreasing and summable. As a result, there exists a decreasing and summable sequence  $r_2(k)$  such that

$$\frac{1}{\sqrt{k}} g_2(k) \leq r_2(k). \quad (7.58)$$



We now turn to the remainder term in (7.48) and show that the sequence  $\frac{1}{k^3}\mathbb{E}\{|S_k + a_k + \eta|^3; A_k\}$  also possesses a decreasing and integrable majorant. It is clear from the definition of  $A_k$ , see (7.27), that

$$|S_k + a_k + \eta| < \sqrt{k}s(\sqrt{k}) \quad \text{on the set } A_k.$$

for all sufficiently large  $k$ . Consequently,

$$|S_k| < \sqrt{k}s(\sqrt{k}) + |a_k| + |\eta| < 2\sqrt{k}s(\sqrt{k}) + |\eta| \quad \text{on the set } A_k.$$

Applying these bounds, we obtain

$$\begin{aligned} \mathbb{E}\{|S_k + a_k + \eta|^3; A_k\} &\leq \mathbb{E}\{|S_k + a_k + \eta|^3; |S_k| \leq 2\sqrt{k}s(\sqrt{k}) + |\eta|; |\eta| \leq \sqrt{k}s(\sqrt{k})\} \\ &\quad + (\sqrt{k}s(\sqrt{k}))^3 \mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})\} \\ &\leq 4\mathbb{E}\{|S_k|^3; |S_k| \leq \sqrt{k}s(\sqrt{k})\} + 16\mathbb{E}\{|\eta|^3; |\eta| \leq \sqrt{k}s(\sqrt{k})\} \\ &\quad + 16|a_k|^3 + (\sqrt{k}s(\sqrt{k}))^3 \mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})\}. \end{aligned}$$

The sequence  $|a_k|$  is bounded. Moreover, the assumption  $\mathbb{E}|\eta| < \infty$  implies, that  $\mathbb{P}\{|\eta| > x\}$  possesses a regularly varying, integrable majorant  $f(x)$ . Then, the function  $x^{-3}\mathbb{E}\{|\eta|^3; |\eta| \leq x\}$  is  $O(f(x))$ . Therefore, the sequences  $k^{-3}\mathbb{E}\{|\eta|^3; |\eta| \leq \sqrt{k}s(\sqrt{k})\}$  is bounded from above by a decreasing summable sequence. The assumption  $\mathbb{E}|\eta| < \infty$  implies also that the decreasing sequence  $(s(\sqrt{k})/\sqrt{k})^3 \mathbb{P}\{|\eta| > \sqrt{k}s(\sqrt{k})\}$  is summable. Thus it remains to show that  $k^{-3}\mathbb{E}\{|S_k|^3; |S_k| \leq \sqrt{k}s(\sqrt{k})\}$  possesses a decreasing and summable majorant. Since  $s(x) = o(x)$ , it suffices to prove that property for  $k^{-3}\mathbb{E}\{|S_k|^3; |S_k| \leq k\}$ . Using (7.37) with  $t = 1$  (7.45), we obtain

$$\begin{aligned} \frac{1}{k^3}\mathbb{E}\{|S_k|^3; |S_k| \leq k\} &\leq \frac{C}{k^{3/2}} + 8\frac{1}{k^2}\mathbb{E}\{\zeta^3(k); \zeta(k) \leq k\} + 8k\mathbb{P}\{\zeta(k) > k/2\} \\ &\leq \frac{C}{k^{3/2}} + 8v_k + 8k\mathbb{P}\{\zeta(k) > k/2\}. \end{aligned}$$

According to (7.41),  $k\mathbb{P}\{\zeta(k) > k/2\} \leq q_k$ . Thus,

$$\frac{1}{k^3}\mathbb{E}\{|S_k + a_k + \eta|^3; A_k\} \leq r_3(k) \tag{7.59}$$

for some decreasing and summable sequence  $r_3(k)$ .

Combining (7.48), (7.50), (7.57), we conclude that

$$\mathbb{E}\{\xi(\sqrt{k})\} = \frac{a - \sigma^2/4 + \mathbb{E}\eta}{2\sqrt{k}} + O(g(k))$$

where

$$g(k) := g_1(k) + g_2(k) + \frac{1}{k^{5/2}}\mathbb{E}\{|S_k + a_k + \eta|^3; A_k\}$$

Furthermore, it follows from (7.51), (7.58) and (7.59) that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} g(k) < \infty.$$

This yields the integrability of the function  $|m_1^{[s(x)]}(x) - (a - \sigma^2/4 + \mathbb{E}\eta)/2x|$ . Thus, (7.46) is proven.

In order to prove (7.47) we first note that this claim will follow from

$$\mathbb{E}\{\xi^2(\sqrt{k}); A_k\} = \frac{\sigma^2}{4} + O(kh_k) \quad (7.60)$$

with some decreasing and summable  $h_k$ .

Applying Taylor's theorem to the function  $(\sqrt{1+u} - 1)^2$  one can easily get

$$\xi^2(\sqrt{k}) = \frac{(S_k + \eta + a_k)^2}{4k} + \tilde{\theta} \frac{(S_k + \eta + a_k)^3}{k^2},$$

where  $\tilde{\theta} = \tilde{\theta}(S_k, \eta, a_k) \in (-1, 1)$  on the set  $A_k$ . From this representation and (7.58) we conclude that

$$\begin{aligned} \mathbb{E}\{\xi^2(\sqrt{k}); A_k\} &= \frac{\mathbb{E}\{(S_k + \eta + a_k)^2; A_k\}}{4k} + O\left(\frac{\mathbb{E}\{(S_k + \eta + a_k)^3; A_k\}}{k^2}\right) \\ &= \frac{\sigma^2}{4} + O\left(\sqrt{k}g_2(k) + \frac{\mathbb{E}\{(S_k + \eta + a_k)^3; A_k\}}{k^2}\right). \end{aligned}$$

It follows from (7.58) and (7.59) that  $\sqrt{k}g_2(k) + \frac{\mathbb{E}\{(S_k + \eta + a_k)^3; A_k\}}{k^2}$  is bounded from above by a decreasing and summable sequence. Thus, the proof of (7.47) is complete.  $\square$

**Corollary 7.9.** *Assume that  $\mathbb{E}|\eta| \log(1 + |\eta|) < \infty$  and that (7.43), (7.44) hold. If the sequence  $\{\zeta(k)\}$  possesses a majorant  $\Xi$  with  $\mathbb{E}\Xi^2 \log(1 + \Xi) < \infty$  then (7.46) and (7.47) are valid.*

*Proof.* We need to show that (7.41) and (7.45) are valid under the assumptions of the corollary. It is clear that

$$\mathbb{E}\{\zeta^2(k); \zeta(k) > k\varepsilon(k)\} \leq \mathbb{E}\{\Xi^2; \Xi > k\varepsilon(k)\}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E}\{\Xi^2; \Xi > k\varepsilon(k)\} = \mathbb{E}\left\{\Xi^2 \left(\sum_{k:k\varepsilon(k) < \Xi} \frac{1}{k}\right)\right\}.$$

Due to the assumption  $\mathbb{E}\Xi^2 \log(1 + \Xi) < \infty$  we can choose  $\varepsilon(k) \rightarrow 0$  such that  $\mathbb{E}\left\{\Xi^2 \left(\sum_{k:k\varepsilon(k) < \Xi} \frac{1}{k}\right)\right\}$  is also finite. Thus, (7.41) holds.

Next, using integration by parts, we get

$$\mathbb{E}\{\zeta^3(k); \zeta(k) \leq k\} \leq 3 \int_0^k x^2 \mathbb{P}\{\zeta(k) \geq x\} dx \leq 3 \int_0^k x^2 \mathbb{P}\{\Xi \geq x\} dx.$$

Set  $\Theta = \Xi^2$ . The finiteness of  $\mathbb{E}\Xi^2$  implies that the function  $\mathbb{P}\{\Theta \geq z\}$  is integrable. Then, by the main result from [16],  $\mathbb{P}\{\Theta \geq z\} \leq f(z)$ , where  $f(z)$  is an integrable, regularly varying with index  $-1$  function. Therefore,

$$\frac{1}{k^2} \mathbb{E}\{\zeta^3(k); \zeta(k) \leq k\} \leq \frac{3}{k^2} \int_0^k x^2 \mathbb{P}\{\Theta \geq x^2\} dx = \frac{3}{2k^2} \int_0^{k^2} \sqrt{z} f(z) dz.$$

The sequence on the right hand side is regularly varying with index  $-1$ . Thus, we have a decreasing majorant for  $k^{-2}\mathbb{E}\{\zeta^3(k); \zeta(k) \leq k\}$ . Therefore, it remains to show that this majorant is summable. By the integraion by parts,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{k^2} \sqrt{z} f(z) dz &= \int_0^{\infty} \sqrt{z} f(z) \left( \sum_{k:k > \sqrt{z}} k^{-2} \right) dz \\ &\leq C \int_0^{\infty} f(z) dz < \infty. \end{aligned}$$

This completes the proof of the corollary.  $\square$

Under the conditions of Proposition (7.8) we have

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = - \left( \frac{\sigma^2/4 - a - \mathbb{E}\eta}{\sigma^2/4} \right) \frac{1}{x} + o(p(x)).$$

This means that (5.3) holds with

$$r(x) = \frac{\rho - 1}{x},$$

where

$$\rho = \frac{\sigma^2/2 - a - \mathbb{E}\eta}{\sigma^2/4}.$$

**Theorem 7.10.** *Assume that all conditions of Proposition 7.8 are valid and that  $a + \mathbb{E}\eta < \sigma^2/2$ . If  $\rho > 2$  we assume that  $\mathbb{E}\{\eta^{\rho/2}; \eta > 0\}$  is finite, and if  $\rho > 4$  then we assume also that there exists a decreasing summable sequence  $u_k$  such that*

$$\frac{1}{k^{\rho/2-1}} \mathbb{E}\{\zeta^{\rho/2}(k); \zeta(k) > k\} \leq u_k. \quad (7.61)$$

Then, for each starting state  $z$ ,

$$\mathbb{P}_z\{\min_{k \leq n} Z_k > z_*\} \sim \frac{c(z)}{n^{\rho/2}} \quad (7.62)$$

and

$$\mathbb{P}_z\left\{\frac{2Z_n}{n\sigma^2} > u \mid \min_{k \leq n} Z_k > z_*\right\} \rightarrow e^{-u}, \quad u > 0, \quad (7.63)$$

where  $z_*$  is the minimal accessible state of  $Z_n$ .

It is easy to see that if  $\mathbb{P}\{\zeta(k) = 0\} > 0$  for all  $k$  and  $\mathbf{P}(\eta \leq 0) > 0$  then  $z_* = 0$ . Furthermore,  $\mathbb{P}\{\eta \leq 0\} = 1$  then 0 is absorbing and we have typical for branching processes statements:

$$\mathbb{P}_z\{Z_n > 0\} \sim \frac{c(z)}{n^{\rho/2}}$$

and

$$\mathbb{P}_z\left\{\frac{2Z_n}{n\sigma^2} > u \mid Z_n > 0\right\} \rightarrow e^{-u}.$$

*Proof of Theorem 7.10.* In order to apply our results from Subsection 5.3 we have to check that conditions (5.8), (5.9) and (5.10) are valid under the assumptions of Theorem 7.10. Since  $r(x) = (\rho-1)/x$ , the function  $U(x)$  is asymptotically equivalent to  $cx^\rho$  with some positive constant  $c$ . Thus, we can replace  $U(x)$  with  $x^\rho$  in (5.9) and (5.10).

We start with (5.10). It follows from the bound

$$|\xi(\sqrt{k})| \leq \frac{|S_k + \eta + a_k|}{\sqrt{k}}$$

that

$$\mathbb{E}\{|\xi(\sqrt{k})|^3; |\xi(\sqrt{k})| \leq s(\sqrt{k})\} \leq \frac{\mathbb{E}\{|S_k + \eta + a_k|^3; A_k\}}{k^{3/2}}.$$

Taking into account (7.59), we get

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \mathbb{E}\{|\xi(\sqrt{k})|^3; |\xi(\sqrt{k})| \leq s(\sqrt{k})\} < \infty.$$

This implies (5.10), since the integrability of a monotone function  $p$  follows from the summability of the sequence  $p(\sqrt{k})/\sqrt{k}$ .

By the same observation, (5.9) will follow from

$$\frac{1}{k^{\rho/2}} \mathbb{E}\{(\sqrt{k} + \xi(\sqrt{k}))^\rho; \xi(\sqrt{k}) > \xi(\sqrt{k})\} \leq w_k, \quad (7.64)$$

where  $w_k$  is decreasing and summable.

We note that

$$\begin{aligned} & \frac{1}{k^{\rho/2}} \mathbb{E}\{(\sqrt{k} + \xi(\sqrt{k}))^\rho; \xi(\sqrt{k}) > \xi(\sqrt{k})\} \\ & \leq \frac{1}{k^{\rho/2}} \mathbb{E}\{(k + S_k + \eta + a_k)^{\rho/2}; S_k + \eta + a_k > 2\sqrt{k}s(\sqrt{k})\} \\ & \leq 3^{\rho/2} \left( \mathbb{P}\{\eta + a_k > \sqrt{k}s(\sqrt{k})\} + \mathbb{P}\{S_k > \sqrt{k}s(\sqrt{k})\} \right) \\ & \quad + \frac{2^{\rho/2}}{k^{\rho/2}} \mathbb{E}\{(S_k + \eta + a_k)^{\rho/2}; S_k + \eta + a_k > 2k\} \\ & \leq 3^{\rho/2} \left( \mathbb{P}\{\eta + a_k > \sqrt{k}s(\sqrt{k})\} + \mathbb{P}\{S_k > \sqrt{k}s(\sqrt{k})\} \right) \\ & \quad + \frac{2^\rho}{k^{\rho/2}} \mathbb{E}\{(S_k)^{\rho/2}; S_k > k\} + \frac{2^\rho}{k^{\rho/2}} \mathbb{E}\{(\eta + a_k)^{\rho/2}; \eta + a_k > k\}. \end{aligned}$$

The summability of the probability terms follows from the assumption  $\mathbb{E}|\eta| < \infty$  and (7.42). Thus, it remains to show that the expectation terms are summable as well. Assume that  $\rho > 2$  and consider  $\mathbb{E}\{(\eta + a_k)^{\rho/2}; \eta + a_k > k\}$ . First we note that this sequence is bounded by  $\mathbb{E}\{(|\eta| + \bar{a})^{\rho/2}; |\eta| + \bar{a} > k\}$ , where  $\bar{a} = \max |a_k|$ . This sequence decreases. Moreover, interchanging the sum and the expectation, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^{\rho/2}} \mathbb{E}\{(|\eta| + \bar{a})^{\rho/2}; |\eta| + \bar{a} > k\} & \leq \mathbb{E} \left\{ (\eta + \bar{a})^\rho \sum_{k=1}^{\eta + \bar{a}} k^{-\rho/2}; \eta + \bar{a} > 0 \right\} \\ & \leq C(\rho) \mathbb{E}\{(\eta + \bar{a})^\rho; \eta + \bar{a} > 0\}. \end{aligned}$$

And the latter expectation is finite due to the assumption  $\mathbb{E}\{\eta^\rho; \eta > 0\} < \infty$ . By the same argument one can easily show that if  $\rho \leq 2$  then the desired summability follows from the finiteness of  $\mathbb{E}|\eta| \log(1 + |\eta|)$ .

Using (7.22) with  $r = 1 + \rho/2$ , we have

$$\begin{aligned} \mathbb{E}\left\{S_k^{\rho/2}; S_k > k\right\} &= k^{\rho/2}\mathbb{P}\{S_k > k\} + \frac{\rho}{2} \int_k^\infty x^{\rho/2-1}\mathbb{P}\{S_k > x\}dx \\ &\leq C(1 + \rho/2) \left( k^{-1} + \frac{\rho}{2} k^{\rho/2+1} \int_k^\infty x^{-\rho/2-2}dx \right) \\ &\quad + k \left( k^{\rho/2}\mathbb{P}\{\zeta(k) > 2k/(\rho+2)\} + k \frac{\rho}{2} \int_k^\infty x^{\rho/2-1}\mathbb{P}\{\zeta(k) > 2x/(\rho+2)\}dx \right) \\ &\leq \frac{2C(1 + \rho/2)}{k} + (1 + \rho/2)^{\rho/2} k \mathbb{E}\{\zeta^{\rho/2}(k); \zeta(k) > 2k/(\rho+2)\}. \end{aligned}$$

Note now that for  $\rho > 4$  one has

$$\begin{aligned} \mathbb{E}\{\zeta^{\rho/2}(k); \zeta(k) > 2k/(\rho+2)\} \\ \leq k^{\rho/2-2} \mathbb{E}\{\zeta^2(k); \zeta(k) > 2k/(\rho+2)\} + \mathbb{E}\{\zeta^{\rho/2}(k); \zeta(k) > 2\}. \end{aligned}$$

Then, it follows from (7.41) and (7.61) that  $k^{-\rho/2}\mathbb{E}\left\{S_k^{\rho/2}; S_k > k\right\}$  possesses a decreasing integrable majorant. This completes the proof of (7.64) in the case when  $\rho > 4$ . The case  $\rho \leq 4$  is even simpler: it suffices to apply the Markov inequality and to use (7.41).

For all sufficiently large  $k$  we have

$$\begin{aligned} \mathbb{P}\{\xi(\sqrt{k}) \leq -s(\sqrt{k})\} &\leq \mathbb{P}\{S_k + \eta + a_k < -\sqrt{k}s(\sqrt{k})\} \\ &\leq \mathbb{P}\{S_k < -\sqrt{k}s(\sqrt{k})/2\} + \mathbb{P}\{\eta + a_k < -\sqrt{k}s(\sqrt{k})/2\}. \end{aligned}$$

The sequence  $\mathbb{P}\{\eta + a_k < -\sqrt{k}s(\sqrt{k})/2\}$  is summable due to  $\mathbb{E}|\eta| < \infty$ , and the summability of  $\mathbb{P}\{S_k < -\sqrt{k}s(\sqrt{k})/2\}$  follows from (7.22) and (7.41). As a result,  $\mathbb{P}\{\xi(\sqrt{k}) \leq -s(\sqrt{k})\}$  is also summable. This implies that (5.8) holds for the chain  $\sqrt{Z_n}$ .

Relations (7.62) and (7.63) follow now from Corollaries 5.16 and 5.18.  $\square$

For size-dependent processes without migration the asymptotic behaviour of the non-extinction probability and the corresponding conditional distribution has been studied earlier by Höpfner [33]. Assumptions in that paper are quite restrictive:  $\mathbb{E}\zeta(k) = 1 + a/k$  with some  $a \in (0, \sigma^2/2]$ ,  $|\sigma^2(k) - \sigma^2| = O(1/k)$  and  $\max_{k \geq 1} \mathbb{E}\zeta^{2+\delta}(k) < \infty$  for some  $\delta > 0$ . If  $a < \sigma^2/2$  then the results in [33] coincide with that in Theorem 7.10, but if  $a = \sigma^2/2$  (this corresponds to  $\rho = 0$ ) then (7.63) is still valid and  $\mathbb{P}\{Z_n > 0\} \sim c/\log n$ . This particular case is not covered by Theorem 7.10.

Zubkov [65] has investigated recurrence times to zero for branching processes with immigration. He has shown that if  $\mathbb{E}\eta < \sigma^2/2$  then there exists a slowly varying function  $L$  such that

$$\mathbb{P}\{\min_{k \leq n} Z_k > 0\} \sim L(n)n^{2\mathbb{E}\eta/\sigma^2-1}.$$

It is also shown there that one can take  $L(n) \equiv C > 0$  if and only if  $\mathbb{E}\eta \log(1+\eta) < \infty$ . Vatutin [63] has shown that (7.63) holds under the same conditions. Zubkov's result shows that the restrictions  $\mathbb{E}|\eta| \log(1 + |\eta|) < \infty$  and (7.41) in Theorem 7.10 are optimal for purely power tail of recurrence times.

Vatutin [62] has initiated the study of branching processes with emigration. More precisely, he has considered sequence  $Y_n$  given by (7.16) with identically distributed  $\zeta(k)$  with mean one and  $\eta \equiv -1$ . For  $\sigma^2 = \mathbb{E}(\zeta - 1)^2 > 2$  he has proven that  $\mathbb{P}\{Y_n > 0 | Y_0 = m\} \sim L_m(n)n^{-1-2/\sigma^2}$  and that  $L_m(n) \equiv c_m > 0$  if and only if  $\mathbb{E}\zeta^2 \log(1 + \zeta) < \infty$ . Moreover, for  $\sigma^2 < 2$  he has shown that  $\mathbb{P}\{Y_n > 0 | Y_0 = m\} \sim c_m n^{-1-2/\sigma^2}$  if and only if  $\mathbb{E}\zeta^{1+2/\sigma^2} < \infty$ . Finally, assuming that all moments of  $\zeta$  are finite, he has proved that  $2Y_n/n\sigma^2$  conditioned on non-extinction converges weakly to the standard exponential distribution. Kaverin [36] has generalized this results to all processes  $Y_n$  satisfying  $\mathbb{E}(-\eta)^{[2+2/\sigma^2]} < \infty$ ,  $\mathbb{E}\zeta^{1+2/\sigma^2} < \infty$  in the case  $\sigma^2 < 2$  and  $\mathbb{E}\zeta^2 \log(1 + \zeta) < \infty$  in the case  $\sigma^2 = 2$ . Specialising Theorem 7.10 to identically distributed  $\zeta(k)$  and non-positive  $\eta$ , we conclude that (7.62) and (7.63) hold for all processes  $Z_n$  satisfying  $\mathbb{E}(-\eta) < \infty$ ,  $\mathbb{E}\zeta^2 \log(1 + \zeta) < \infty$  and  $\mathbb{E}\zeta^{1+2/\sigma^2} < \infty$  in the case  $\sigma^2 < 2$ . We see that our restrictions on the emigration component  $\eta$  are much weaker than in [36].

Processes  $Z_n$  and  $Y_n$  are formally different. But it is intuitively clear that the difference between their definitions should have no influence on the asymptotic behaviour. And in the case of identically distributed  $\zeta(k)$  and nonpositive  $\eta$  one can transfer asymptotics for one process into corresponding asymptotics for another one. Indeed, if we define

$$W_{2k+1} := (W_{2k} + \eta_{k+1})^+, \quad W_{2k+2} = \sum_{i=1}^{W_{2k+1}} \zeta_{k+1,i}, \quad k \geq 0,$$

then  $Y_0 = W_0 = m$  implies that  $Y_n = W_{2n}$  and  $Z_n = W_{2n+1}$  with  $Z_0 = (m + \eta_1)^+$ . In the case of emigration processes ( $\mathbb{P}\{\eta \leq 0\} = 1$ ) we have obviously that the sequence  $\{W_k > 0\}$  is monotone decreasing. If (7.62) is valid for every fixed starting point  $Z_0$  then it is also valid for  $Z_0 = (m + \eta)^+$ . As a result, we have

$$\mathbb{P}\{Y_n > 0 | Y_0 = m\} \sim \sum_{j=1}^m \mathbb{P}\{m + \eta = j\} \mathbb{P}\{Z_n > 0 | Z_0 = j\} \sim c(m)n^{-\rho/2}.$$

Furthermore, recalling that  $Y_n = \sum_{i=1}^{Z_{n-1}} \zeta_{n,i}$  and applying the law of large numbers, it is easy to see that (7.63) yields the same limiting behaviour of  $Y_n$ .

Kosygina and Mountford [47] have proven (7.62) for a special model of branching processes with migration. This model appears in the description of exited random walks on integers.

If  $\mathbb{P}\{\eta > 0\}$  is positive then 0 is not absorbing and, consequently,  $Z_n$  is irreducible. Then we can apply Theorem 5.1 to  $\sqrt{Z_n}$  and derive the tail behaviour of the stationary measure of  $Z_n$ : for any constants  $a < b$  we have

$$\pi_Z(ak, bk) \sim C \int_{\sqrt{ak}}^{\sqrt{bk}} y^{1-\rho} dy \quad \text{as } k \rightarrow \infty.$$

It follows from Theorem 7.10 that  $Z_n$  is positive recurrent when  $\rho > 2$ . In this case may apply also Theorem 5.20 and obtain tail asymptotics for  $Z_n$ .

If  $\rho \in (0, 2)$  then the pre-limiting behaviour of  $Z_n$  is described in Theorem 5.21. If  $\rho = 2$  then, according to (7.62),  $Z_n$  is also null-recurrent but its behaviour is not covered by Theorem 5.21. Here we can apply Theorem 5.22. Since  $G(x) \sim \log x$  under the assumptions of Theorem 7.10, we conclude that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \frac{\log Z_n}{\log n} \leq x \right\} = x, \quad x \in [0, 1]. \quad (7.65)$$

First result of this type has been obtained by Foster [27] for a critical Galton-Watson process with immigration at zero. Formally, we can not say that Foster's result follows from (7.65). But since all calculations we have made in the proof of Theorem 7.10 are valid for processes without migration, it is easy to see that adding immigration at zero does not change the asymptotic behaviour of truncated moments. Therefore, Theorem 5.22 is applicable to the process from [27] if the number of immigrating individuals has finite mean.

Nagaev and Khan [58] have proven (7.65) for a critical process with migration. More precisely, they have considered the sequence  $Y_n$  given by (7.16) with identically distributed  $\zeta(k)$  with mean one and finite variance. Let us compare our moment assumptions with that in [58]. First we note that if  $\zeta(k)$  are identically distributed and have finite variance then (7.43)–(7.45) hold automatically. Moreover, (7.41) is equivalent to the assumption  $\mathbb{E}\zeta^2 \log(1+|\zeta|) < \infty$ . This is a bit more restrictive than the second moment assumption in [58]. Further, we have assumed that  $\mathbb{E}|\eta| \log(1+|\eta|)$  is finite, which is weaker than the corresponding condition in [58]. There is assumed that  $\mathbb{E}\eta^2 < \infty$  and  $\mathbb{P}\{\eta > -m\} = 1$  for some  $m \geq 1$ .

Comparing our theorems with the known in the literature results for branching processes with migration, we conclude that the only weakness of the transformation  $\sqrt{Z_n}$  is the fact that it is not clear how to deal with the case when one has tail asymptotics with non-trivial slowly varying functions. Recall that the only obstacle is to show (5.3) in the case when  $2m_1^{[s(x)]}(x)/m_2^{[s(x)]}(x) - c/x$  is not integrable for any constant  $c$ .

## 7.4 Cramér–Lundberg risk processes with level-dependent premium rate

In context of the collective theory of risk, the classical *Cramér–Lundberg model* is defined as follows. An insurance company receives the constant inflow of premium at rate  $c$ , that is, the premium income is assumed to be linear in time with rate  $v$ . It is also assumed that the claims incurred by the insurance company arrive according to a homogeneous renewal process  $N_t$  with intensity  $\lambda$  and the sizes (amounts)  $\xi_n \geq 0$  of the claims are independent identically distributed random variables with common mean  $b$ . The  $\xi$ 's are assumed to be independent of the process  $N_t$ . The company has an initial risk reserve  $x = R_0 \geq 0$ .

Then the risk reserve  $R_t$  at time  $t$  is equal to

$$R_t = x + vt - \sum_{i=1}^{N_t} \xi_i.$$

The probability

$$\mathbb{P}\{R_t \geq 0 \text{ for all } t \geq 0\} = \mathbb{P}\left\{\min_{t \geq 0} R_t \geq 0\right\}$$

is the probability of ultimate survival and

$$\begin{aligned} \psi(x) &:= \mathbb{P}\{R_t < 0 \text{ for some } t \geq 0\} \\ &= \mathbb{P}\left\{\min_{t \geq 0} R_t < 0\right\} \end{aligned}$$

is the probability of ruin. We have

$$\psi(x) = \mathbb{P}\left\{\sum_{i=1}^{N_t} \xi_i - vt > x \text{ for some } t \geq 0\right\}.$$

Since  $v > 0$ , the ruin can only occur at a claim epoch. Therefore,

$$\psi(x) = \mathbb{P}\left\{\sum_{i=1}^n \xi_i - vT_n > x \text{ for some } n \geq 1\right\},$$

where  $T_n$  is the  $n$ th claim epoch, so that  $T_n = \tau_1 + \dots + \tau_n$  where the  $\tau$ 's are independent identically distributed random variables with common mean  $1/\lambda$ , so that  $N(t) := \max\{n \geq 1 : T_n \leq t\}$ . Denote  $X_i := \xi_i - v\tau_i$  and  $S_n := X_1 + \dots + X_n$ , then

$$\psi(x) = \mathbb{P}\left\{\sup_{n \geq 1} S_n > x\right\}.$$

This relation represents the ruin probability problem as the tail probability problem for the maximum of the associated random walk  $S_n$ . Let the *net-profit condition*

$$v > v_c := \mathbb{E}\xi_1 / \mathbb{E}\tau_1 = \lambda \mathbb{E}\xi_1$$

hold, thus  $S_n$  has a negative drift by the strong law of large numbers and  $\psi(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

In this section we consider a risk process where the premium  $v(y)$  depends on the current level of risk reserve  $R(t) = y$ , so  $R(t)$  satisfies the equality

$$R(t) = \int_0^t v(R(s))ds - \sum_{j=1}^{N(t)} \xi_j; \quad (7.66)$$



$v(y)$  is assumed to be a measurable positive function. The probability of ruin given initial risk reserve  $x$  is again denoted by  $\psi(x)$ . We consider the case where  $v(y)$  approaches the critical value  $v_c$  at infinity, that is,

$$v(y) \rightarrow v_c \quad \text{as } y \rightarrow \infty. \quad (7.67)$$

To the best of our knowledge, the only case where  $\psi(x)$  is explicitly calculable is the case of exponentially distributed  $\tau_1$  and  $\xi_1$ , say with parameters  $\lambda$  and  $\mu$  respectively, so hence  $v_c = \lambda/\mu$ . In this case, for some  $c_0 > 0$ ,

$$\begin{aligned} \psi(x) &= c_0 \int_x^\infty \frac{1}{v(y)} \exp\left\{-\mu y + \lambda \int_0^y \frac{dz}{v(z)}\right\} dy \\ &= c_0 \int_x^\infty \frac{1}{v(y)} \exp\left\{\lambda \int_0^y \left(\frac{1}{v(z)} - \frac{1}{v_c}\right) dz\right\} dy, \end{aligned} \quad (7.68)$$

see, e.g. Corollary 1.9 in Albrecher and Asmussen [1, Ch. VIII]. Then, by (7.67),

$$\psi(x) \sim \frac{c_0}{v_c} \int_x^\infty \exp\left\{\lambda \int_0^y \left(\frac{1}{v(z)} - \frac{1}{v_c}\right) dz\right\} dy \quad \text{as } x \rightarrow \infty.$$

If the premium rate  $v(z)$  approaches  $v_c$  at a rate of  $\theta/z$ ,  $\theta > 0$ , more precisely, if

$$\left|v(z) - v_c - \frac{\theta}{z}\right| \leq p(z) \quad \text{for all } z > 1, \quad (7.69)$$

where  $p(z) > 0$  is integrable at infinity decreasing function, then we get

$$\frac{1}{v(z)} = \frac{1}{v_c} - \frac{\theta}{v_c^2 z} + O(p(z) + z^{-2})$$

and consequently

$$\lambda \int_0^y \left(\frac{1}{v(z)} - \frac{1}{v_c}\right) dz = -\frac{\theta\mu^2}{\lambda} \log y + c_1 + o(1) \quad \text{as } y \rightarrow \infty,$$

where  $c_1$  is a finite number. Then, for  $C := c_0 e^{c_1}/(\theta\mu - \lambda/\mu) > 0$ ,

$$\psi(x) \sim \frac{C}{x^{\theta\mu^2/\lambda-1}} \quad \text{as } x \rightarrow \infty. \quad (7.70)$$

A similar asymptotic expression can be obtained also in the case where Laplace transforms of variables  $\xi_1$  and  $\tau_1$  are rational functions, see Albrecher et al. [2].

If the premium rate  $v(z)$  approaches  $v_c$  at a rate of  $\theta/z^\alpha$ ,  $\theta > 0$  and  $\alpha \in (0, 1)$ , more precisely, if

$$\left|v(z) - v_c - \frac{\theta}{z^\alpha}\right| \leq p(z) \quad \text{for all } z > 1, \quad (7.71)$$

where  $p(z) > 0$  is integrable at infinity decreasing function, then we get

$$\frac{1}{v(z)} = \frac{1}{v_c} \sum_{j=0}^{\infty} (-1)^j \left(\frac{\theta}{v_c}\right)^j \frac{1}{z^{\alpha j}} + O(p(z))$$

Let  $\gamma := \min\{k \in \mathbb{N} : k\alpha > 1\}$ . Then

$$\frac{1}{v(z)} = \frac{1}{v_c} \sum_{j=0}^{\gamma-1} (-1)^j \left(\frac{\theta}{v_c}\right)^j \frac{1}{z^{\alpha j}} + O(p_1(z))$$

where  $p_1(x) = p(z) + z^{-\gamma\alpha}$  is integrable at infinity. Consequently,

$$\begin{aligned} \lambda \int_0^y \left(\frac{1}{v(z)} - \frac{1}{v_c}\right) dz &= \frac{\lambda}{v_c} \int_1^y \sum_{j=1}^{\gamma-1} (-1)^j \left(\frac{\theta}{v_c}\right)^j \frac{1}{z^{\alpha j}} dz + c_2 + o(1) \\ &= \frac{\lambda}{v_c} \sum_{j=1}^{\gamma-1} (-1)^j \left(\frac{\theta}{v_c}\right)^j \frac{y^{1-\alpha j}}{1-\alpha j} + c_3 + o(1) \quad \text{as } y \rightarrow \infty, \end{aligned}$$

where  $c_3$  is a finite number because  $p_1(x)$  is integrable.

Let, for example,  $\alpha \in (1/2, 1)$ . Then

$$\lambda \int_0^y \left(\frac{1}{v(z)} - \frac{1}{v_c}\right) dz = -\frac{\theta\mu^2}{\lambda(1-\alpha)} y^{1-\alpha} + c_3 + o(1) \quad \text{as } y \rightarrow \infty.$$

Therefore, for  $C_1 := c_0 e^{c_2}/\theta\mu > 0$  and  $C_2 := \theta\mu^2/\lambda(1-\alpha) > 0$ ,

$$\psi(x) \sim C_1 x^\alpha e^{-C_2 x^{1-\alpha}} \quad \text{as } x \rightarrow \infty. \quad (7.72)$$

Let us extend these results for not necessarily exponential distributions where there are no formulas like (7.68) for  $\psi(x)$  available. In that case we can only derive lower and upper bounds for  $\psi(x)$ .

#### 7.4.1 Approaching critical premium rate at a rate of $\theta/x$

Recall that the ruin can only occur at a claim epoch. Therefore, the ruin probability may be reduced to that for the embedded Markov chain  $R_n := R(T_n)$ ,  $n \geq 1$ ,  $R_0 := x$ , that is,

$$\psi(x) = \mathbb{P}\{R_n < 0 \text{ for some } n \geq 0\}.$$

Denote jumps of the chain  $R_n$  by  $\xi(x)$  and by  $m_k^{[s(x)]}(x)$  its  $k$ th truncated moments.

**Proposition 7.11.** *Assume (7.69) and that both  $\mathbb{E}\tau_1^2 \log(1+\tau_1)$  and  $\mathbb{E}\xi_1^2 \log(1+\xi_1)$  are finite. Then there exists an increasing function  $s(x) = o(x)$  such that*

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} = \frac{\rho + 1}{x} + O(p_1(x)),$$

for some decreasing integrable function  $p_1(x)$ , where

$$\rho := \frac{2\theta\mathbb{E}\tau_1}{\mathbb{V}\text{ar}\xi_1 + v_c^2\mathbb{V}\text{ar}\tau_1} - 1.$$

*Proof.* The dynamic of the risk reserve between two consequent claims is governed by the differential equation  $R'(t) = v(R(t))$ . Let  $V_x(t)$  denote its solution with the initial value  $x$ , so then

$$V_x(t) = x + \int_0^t v(V_x(s))ds.$$

By (7.69),

$$\begin{aligned} v(y) &\leq v_c + \theta/y + p(y) \\ &\leq v_c + \theta/x + p(x) \quad \text{for all } y \geq x, \end{aligned}$$

therefore

$$V_x(t) - x \leq v_c t + \theta t/x + p(x)t, \quad t > 0. \quad (7.73)$$

On the other hand, again by (7.69),

$$\begin{aligned} v(y) &\geq v_c + \theta/y - p(y) \\ &\geq v_c + \theta/y - p(x) \quad \text{for all } y \geq x, \end{aligned}$$

Hence,

$$\begin{aligned} V_x(t) - x &\geq v_c t + \theta \int_0^t \frac{ds}{V_x(s)} - p(x)t \\ &\geq v_c t + \theta \int_0^t \frac{ds}{x + (v_c + \theta/x + p(x))s} - p(x)t \\ &= v_c t + \frac{\theta}{v_c + \theta/x + p(x)} \log(1 + (v_c + \theta/x + p(x))t/x) - p(x)t, \end{aligned}$$

where the second inequality follows from (7.73). Therefore,

$$V_x(t) - x \geq v_c t + \frac{\theta}{v_c + \theta/x + p(x)} \log(1 + v_c t/x) - p(x)t, \quad (7.74)$$

Since  $\xi(x) = V_x(\tau_1) - x - \xi_1$ , it follows from (7.73) and (7.74) that

$$v_c \tau_1 - \xi_1 + \frac{\theta}{v_c + \theta/x + p(x)} \log\left(1 + \frac{v_c \tau_1}{x}\right) \leq \xi(x) \leq v_c \tau_1 - \xi_1 + \frac{\theta \tau_1}{x} + p(x)\tau_1. \quad (7.75)$$

Recalling that  $v_c = \mathbb{E}\tau_1/\mathbb{E}\xi_1$ , we get

$$\frac{\theta}{v_c + \theta/x + p(x)} \mathbb{E} \log\left(1 + \frac{v_c \tau_1}{x}\right) \leq m_1(x) \leq \frac{\theta}{x} \mathbb{E}\tau_1 + p(x)\mathbb{E}\tau_1.$$

By the inequality  $\log(1+z) \geq z - z^2/2$ ,

$$\mathbb{E} \log\left(1 + \frac{v_c \tau_1}{x}\right) \geq \frac{v_c \mathbb{E}\tau_1}{x} - \frac{v_c^2 \mathbb{E}\tau_1^2}{2x^2}.$$

Therefore,

$$m_1(x) = \frac{\theta \mathbb{E}\tau_1}{x} + O(p(x) + 1/x^2). \quad (7.76)$$

From this expression we have

$$\begin{aligned} m_2(x) &= \mathbb{V}ar(V_x(\tau_1) - x - \xi_1) + O(p^2(x) + 1/x^2) \\ &= \mathbb{V}ar(V_x(\tau_1) - x) + \mathbb{V}ar\xi_1 + O(p^2(x) + 1/x^2). \end{aligned}$$

Recalling that

$$v_c t - p(x)t \leq V_x(t) - x \leq v_c t + \frac{\theta}{x}t + p(x)t,$$

we get

$$(v_c - p(x))\mathbb{E}\tau_1 \leq \mathbb{E}(V_x(\tau_1) - x) \leq (v_c + \theta/x + p(x))\mathbb{E}\tau_1$$

and

$$(v_c - p(x))^2 \mathbb{E}\tau_1^2 \leq \mathbb{E}(V_x(\tau_1) - x)^2 \leq (v_c + \theta/x + p(x))^2 \mathbb{E}\tau_1^2.$$

Hence,

$$\mathbb{V}ar(V_x(\tau_1) - x) = v_c^2 \mathbb{V}ar\tau_1 + O(1/x),$$

which in its turn implies

$$m_2(x) = \mathbb{V}ar\xi_1 + v_c^2 \mathbb{V}ar\tau_1 + O(1/x). \quad (7.77)$$

Together with (7.76) it yields that

$$\frac{2m_1(x)}{m_2(x)} = \frac{2\theta \mathbb{E}\tau_1}{\mathbb{V}ar\xi_1 + v_c^2 \mathbb{V}ar\tau_1} \frac{1}{x} + O(p(x) + 1/x^2).$$

But we need such kind of expansion for truncated moments. For any truncation level  $s(x)$  we have

$$\begin{aligned} &|V_x(\tau_1) - x - \xi_1| \mathbb{I}\{|V_x(\tau_1) - x - \xi_1| > s(x)\} \\ &\leq (V_x(\tau_1) - x + \xi_1) \mathbb{I}\{V_x(\tau_1) - x > s(x) \text{ or } \xi_1 > s(x)\} \\ &\leq (V_x(\tau_1) - x) \mathbb{I}\{V_x(\tau_1) - x > s(x)\} + \xi_1 \mathbb{I}\{\xi_1 > s(x)\} \\ &\quad + \xi_1 \mathbb{I}\{V_x(\tau_1) - x > s(x)\} + (V_x(\tau_1) - x) \mathbb{I}\{\xi_1 > s(x)\}. \end{aligned} \quad (7.78)$$

Since  $V_x(t) - x \leq c_1 t$  for some  $c_1 < \infty$ , we then get

$$\begin{aligned} |m_1(x) - m_1^{[s(x)]}| &\leq \mathbb{E}\{|V_x(\tau_1) - x - \xi_1|; |V_x(\tau_1) - x - \xi_1| > s(x)\} \\ &\leq c_1 \mathbb{E}\{\tau_1; \tau_1 > s(x)/c_1\} + \mathbb{E}\{\xi_1; \xi_1 > s(x)\} \\ &\quad + \mathbb{E}\xi_1 \mathbb{P}\{\tau_1 > s(x)/c_1\} + c_1 \mathbb{E}\tau_1 \mathbb{P}\{\xi_1 > s(x)\}. \end{aligned}$$

It follows from the finiteness of  $\mathbb{E}\tau_1^2$  and  $\mathbb{E}\xi_1^2$  that there exists an increasing function  $s_1(x) = o(x)$  such that both  $\mathbb{E}\{\tau_1; \tau_1 > s_1(x)/c_1\}$  and  $\mathbb{E}\{\xi_1; \xi_1 > s_1(x)\}$  are integrable. Consequently,  $|m_1(x) - m_1^{[s_1(x)]}(x)|$  is bounded by a decreasing integrable function. Combining this with (7.76), we conclude that

$$m_1^{[s_1(x)]}(x) = \frac{\theta \mathbb{E}\tau_1}{x} + o(p_2(x)), \quad (7.79)$$

where  $p_2$  is a decreasing integrable function.

By similar arguments, for some  $c_2 < \infty$ ,

$$\begin{aligned} 0 \leq m_2(x) - m_2^{[s_2(x)]}(x) &\leq c_2 \left( \mathbb{E}\{\tau_1^2; \tau_1 > s(x)/c_1\} + \mathbb{E}\{\xi_1^2; \xi_1 > s(x)\} \right. \\ &\quad \left. + \mathbb{E}\xi_1^2 \mathbb{P}\{\tau_1 > s(x)/c_1\} + \mathbb{E}\tau_1^2 \mathbb{P}\{\xi_1 > s(x)\} \right). \end{aligned}$$

It follows from the finiteness of  $\mathbb{E}\xi_1^2 \log(1 + \xi_1)$  and  $\mathbb{E}\tau_1^2 \log(1 + \tau_1)$  that there exists an increasing function  $s_2(x) = o(x)$  such that both  $x^{-1} \mathbb{E}\{\tau_1^2; \tau_1 > s_2(x)/c_1\}$  and  $x^{-1} \mathbb{E}\{\xi_1^2; \xi_1 > s_2(x)\}$  are integrable at infinity. Then  $(m_2(x) - m_2^{[s_2(x)]}(x))/x$  is integrable too. From this fact and (7.77) we get

$$m_2^{[s_2(x)]}(x) = \text{Var}\xi_1 + v_c^2 \text{Var}\tau_1 + o(xp_3(x)) \quad (7.80)$$

for some decreasing integrable function  $p_3(x)$ . Taking now  $s(x) = \max(s_1(x), s_2(x)) = o(x)$  we conclude from (7.79) and (7.80) the desired result.  $\square$

**Theorem 7.12.** *Assume that both  $\mathbb{E}\xi_1^2$  and  $\mathbb{E}\tau_1^2$  are finite. If*

$$\theta > \frac{\text{Var}\xi_1 + v_c^2 \text{Var}\tau_1}{2\mathbb{E}\tau_1},$$

*then  $R_n$  is transient or, equivalently,  $\psi(x) < 1$  for all  $x > 0$ . Set*

$$\rho = \theta \frac{2\mathbb{E}\tau_1}{\text{Var}\xi_1 + v_c^2 \text{Var}\tau_1} - 1 > 0.$$

*If both  $\mathbb{E}\tau_1^2 \log(1 + \tau_1)$  and  $\mathbb{E}\xi_1^{\rho+2}$  are finite, then there exist positive constants  $c_1$  and  $c_2$  such that*

$$\frac{c_1}{(1+x)^\rho} \leq \psi(x) \leq \frac{c_2}{(1+x)^\rho} \quad \text{for all } x > 0.$$

*Proof.* It follows from (7.77) and (7.79) that

$$\frac{2m_1^{[s(x)]}(x)}{m_2^{[s(x)]}(x)} \geq \frac{2m_1^{[s(x)]}(x)}{m_2(x)} \geq \frac{1+\varepsilon}{x}$$

for some small  $\varepsilon$  and for all  $x \geq x_0(\varepsilon)$ . Furthermore, from the elementary bound  $\mathbb{P}\{\xi(x) < -s(x)\} \leq \mathbb{P}\{\xi_1 > s(x)\}$  and the finiteness of  $\mathbb{E}\xi_1^2$  we infer that  $x\mathbb{P}\{\xi(x) <$

$-s(x)\}$  is integrable for some increasing function  $s(x) = o(x)$ . Thus, all the conditions of Theorem 2.18 are valid and, consequently,  $R_n$  is transient.

The second part of the theorem follows from Theorem 2.20 if it is shown that its conditions hold for  $R_n$  with  $U(x) = c/(x+1)^\rho$  which corresponds to  $r(x) = (\rho+1)/(x+1)$ . For the right tail of  $\xi(x)$  we have

$$\begin{aligned} \mathbb{P}\{\xi(x) > s(x)\} &\leq \mathbb{P}\{V_x(\tau_1) - x > s(x)\} \\ &\leq \mathbb{P}\{\tau_1 > s(x)/c_1\} = o(p(x)/x) \end{aligned}$$

due to the assumption  $\mathbb{E}\tau_1^2 < \infty$ . By the same argument,

$$\begin{aligned} \mathbb{E}\left\{\frac{1}{(1+x+\xi(x))^\rho}; \xi(x) < -s(x)\right\} &\leq \mathbb{E}\left\{\frac{1}{(1+x+\xi)^\rho}; \xi(x) < -s(x)\right\} \\ &\leq \mathbb{P}\{\xi(x) < -s(x)\} \\ &= o(p(x)/x^{\rho+1}) \end{aligned}$$

due to the assumption  $\mathbb{E}\xi^{\rho+2} < \infty$ . Obviously,

$$|\xi(x)| \leq V_x(\tau_1) - x + \xi_1 \leq c_1\tau_1 + \xi_1 =: \Xi$$

Then

$$\begin{aligned} \mathbb{E}\{|\xi(x)|^3; |\xi(x)| \leq s(x)\} &\leq 3 \int_0^{s(x)} y^2 \mathbb{P}\{|\xi(x)| > y\} dy \\ &\leq 3 \int_0^{s(x)} y^2 \mathbb{P}\{\Xi > y\} dy \\ &\leq 3 \int_0^x y \mathbb{E}\{\Xi; \Xi > y\} dy. \end{aligned}$$

Finiteness of  $\mathbb{E}\Xi^2$  implies integrability of  $x^{-2} \int_0^x y \mathbb{E}\{\Xi; \Xi > y\} dy$ . In addition, this function is decreasing because

$$\begin{aligned} \frac{d}{dx} \frac{1}{x^2} \int_0^x y \mathbb{E}\{\Xi; \Xi > y\} dy &= -\frac{2}{x^3} \int_0^x y \mathbb{E}\{\Xi; \Xi > y\} dy + \frac{1}{x} \mathbb{E}\{\Xi; \Xi > x\} \\ &\leq -\frac{2}{x^3} \mathbb{E}\{\Xi; \Xi > x\} \int_0^x y dy + \frac{1}{x} \mathbb{E}\{\Xi; \Xi > x\} \\ &= 0, \end{aligned}$$

and the condition (2.113) is satisfied for  $R_n$ . □

#### 7.4.2 Approaching critical premium rate at a rate of $\theta/x^\alpha$

In this subsection we consider the case (7.71). In order to understand the asymptotic behavior of the ruin probability under this rate of approaching  $v_c$ , we first derive asymptotic estimates for the moments of  $V_x(\tau) - x$ . Define

$$\gamma: = \min\{k \geq 1 : \alpha k > 1\}.$$

**Lemma 7.13.** *Let  $\mathbb{E}\tau^{\gamma+1} < \infty$  and*

$$v_-(x) \leq v(x) \leq v_+(x) \quad \text{for all } x, \quad (7.81)$$

*where both  $v_-(x)$  and  $v_+(x)$  are decreasing functions. Then, for all  $k \leq \gamma$ ,*

$$\mathbb{E}\tau^k v_-(x + \tau v_+(x)) \leq \mathbb{E}(V_x(\tau) - x)^k \leq v_+^k(x) \mathbb{E}\tau^k. \quad (7.82)$$

*If, in addition, (7.71) holds, then there exists an integrable decreasing function  $p_1(x)$  such that, for all  $k \leq \gamma$ ,*

$$\mathbb{E}(V_x(\tau) - x)^k = (v_c + \theta/x^\alpha)^k \mathbb{E}\tau^k + O(p_1(x)) \quad \text{as } x \rightarrow \infty. \quad (7.83)$$

*Proof.* Due to (7.81),  $v(z) \leq v_+(x)$  for all  $z \geq x$ . Hence

$$\begin{aligned} V_x(t) &= x + \int_0^t v(V_x(s)) ds \\ &\leq x + \int_0^t v_+(x) ds = x + tv_+(x), \end{aligned} \quad (7.84)$$

and the inequality on the right hand side of (7.82) follows. It follows from left hand side inequality in (7.81) and from the last upper bound for  $V_x(t)$  that

$$V_x(t) - x \geq \int_0^t v_-(V_x(s)) ds \geq tv_-(x + tv_+(x)), \quad (7.85)$$

and the left hand side bound in (7.82) is proven.

Owing to (7.71),  $v(z)$  is sandwiched between the two eventually decreasing functions  $v_\pm := v_c + \theta/z^\alpha \pm p(z)$ . Therefore, applying the right hand side bound in (7.82) we get

$$\begin{aligned} \mathbb{E}(V_x(\tau) - x)^k &\leq (v_c + \theta/x^\alpha + p(x))^k \mathbb{E}\tau^k \\ &= (v_c + \theta/x^\alpha)^k \mathbb{E}\tau^k + O(p(x)) \quad \text{as } x \rightarrow \infty. \end{aligned} \quad (7.86)$$

From the lower bound in (7.82) we deduce, for all  $k \leq \gamma$ ,

$$\begin{aligned} \mathbb{E}(V_x(\tau) - x)^k &\geq \mathbb{E}\tau^k \left( v_c + \frac{\theta}{(x + \tau(v_c + \theta/x^\alpha + p(x)))^\alpha} - p(x) \right)^k \\ &\geq \mathbb{E}\tau^k \left( v_c + \frac{\theta}{(x + c_1 \tau)^\alpha} \right)^k + O(p(x)) \quad \text{for some } c_1 < \infty. \end{aligned}$$

Hence,

$$\mathbb{E}(V_x(\tau) - x)^k \geq \mathbb{E} \left\{ \tau^k \left( v_c + \frac{\theta}{(x + c_1 \tau)^\alpha} \right)^k ; \tau \leq x \right\} + O(p(x)).$$

By the inequality  $1/(1+y)^\alpha \geq 1 - \alpha y$ , we infer that there exists a constant  $c_2$  such that

$$\frac{1}{(x + c_1 t)^\alpha} \geq \frac{1}{x^\alpha} \left( 1 - \frac{\alpha}{x} c_1 t \right) = \frac{1}{x^\alpha} - c_2 \frac{t}{x^{\alpha+1}} \quad \text{for all } x \geq 1.$$

Therefore,

$$\begin{aligned} \mathbb{E}(V_x(\tau) - x)^k &\geq \mathbb{E}\left\{\tau^k \left(v_c + \frac{\theta}{x^\alpha} - c_2 \frac{\tau}{x^{\alpha+1}}\right)^k; \tau \leq x\right\} + O(p(x)) \\ &\geq (v_c + \theta/x^\alpha)^k \mathbb{E}\{\tau^k; \tau \leq x\} \\ &\quad - c_3 \sum_{j=1}^k \frac{1}{x^{j(\alpha+1)}} \mathbb{E}\{\tau^{k+j}; \tau \leq x\} - c_3 p(x), \end{aligned} \quad (7.87)$$

for some  $c_3 < \infty$ . Then, due to integrability of  $p(x)$ , in order to prove that

$$\mathbb{E}(V_x(\tau) - x)^k \geq (v_c + \theta/x^\alpha)^k \mathbb{E}\tau^k - p_1(x) \quad (7.88)$$

for some decreasing integrable  $p_1(x)$ , it suffices to show that

$$\mathbb{E}\{\tau^k; \tau > x\}$$

and

$$x^{-j(\alpha+1)} \mathbb{E}\{\tau^{k+j}; \tau \leq x\}, \quad 1 \leq j \leq k,$$

are bounded by decreasing integrable at infinity functions. Indeed, the integral of the first function—which decreases itself—equals  $\mathbb{E}\tau^{k+1}$  which is finite due to  $k \leq \gamma$  and finiteness of the  $(\gamma + 1)$ st moment of  $\tau$ . Concerning the second function, first notice that

$$\mathbb{E}\{\tau^{k+j}; \tau \leq x\} \leq x^{j-1} \mathbb{E}\tau^{k+1}, \quad j \geq 1, \quad k \leq \gamma.$$

Thus,

$$x^{-j(\alpha+1)} \mathbb{E}\{\tau^{k+j}; \tau \leq x\} \leq \frac{\mathbb{E}\tau^{\gamma+1}}{x^{1+j\alpha}}, \quad j \geq 1, \quad k \leq \gamma.$$

So, (7.88) is proven which together with (7.86) completes the proof.  $\square$

**Proposition 7.14.** *Assume that (7.71) is valid. If both  $\mathbb{E}\tau_1^{1+\gamma}$  and  $\mathbb{E}\xi_1^{1+\gamma}$  are finite, then there exists  $s(x) = o(x^\alpha)$  such that, for all  $k \leq \gamma$ ,*

$$m_k^{[s(x)]}(x) = \sum_{j=0}^k \frac{a_{k,j}}{x^{\alpha j}} + O(x^{\alpha(k-1)} p_2(x)) \quad \text{as } x \rightarrow \infty,$$

where  $p_2(x)$  is a decreasing integrable at infinity function and

$$a_{k,j} := \theta^j \binom{k}{j} \mathbb{E}\tau_1^j (v_c \tau_1 - \xi_1)^{k-j}, \quad j \leq k \leq \gamma.$$

*Proof.* It follows from the definition of  $\xi(x)$  that

$$\mathbb{E}\xi^k(x) = \mathbb{E}(V_x(\tau_1) - x - \xi_1)^k = \sum_{j=0}^k \binom{k}{j} \mathbb{E}(V_x(\tau_1) - x)^j \mathbb{E}(-\xi_1)^{k-j}.$$



Applying Lemma 7.13, we then obtain

$$\begin{aligned} m_k(x) &:= \mathbb{E}\xi^k(x) = \sum_{j=0}^k \binom{k}{j} \left(v_c + \frac{\theta}{x^\alpha}\right)^j \mathbb{E}\tau_1^j \mathbb{E}(-\xi_1)^{k-j} + O(p_1(x)) \\ &=: \sum_{j=0}^k \frac{a_{k,j}}{x^{\alpha j}} + O(p_1(x)). \end{aligned}$$

Now, in view of (7.78) we have

$$\begin{aligned} |m_k(x) - m_k^{[s(x)]}(x)| &= O\left(\mathbb{E}\{(V_x(\tau_1) - x)^k; V_x(\tau_1) - x > s(x)\} + \mathbb{E}\{\xi_1^k; \xi_1 > s(x)\}\right. \\ &\quad \left.+ \mathbb{E}\xi_1^k \mathbb{P}\{V_x(\tau_1) - x > s(x)\} + \mathbb{E}(V_x(\tau_1) - x)^k \mathbb{P}\{\xi_1 > s(x)\}\right) \\ &= O\left(\mathbb{E}\{\tau_1^k; \tau_1 > s(x)/c_1\} + \mathbb{E}\{\xi_1^k; \xi_1 > s(x)\}\right). \end{aligned}$$

The finiteness of  $\mathbb{E}\tau_1^{\gamma+1}$  and  $\mathbb{E}\xi_1^{\gamma+1}$  implies existence of  $s(x) = o(x^\alpha)$  such that both terms on the right hand side being multiplied by  $x^{-\alpha(k-1)}$  are integrable for all  $k \leq \gamma$ . Thus, the proof is complete.  $\square$

Now we state the main result in this subsection.

**Theorem 7.15.** *Assume that (7.71) is valid. If  $\mathbb{E}\tau_1^{\gamma+1} < \infty$  and  $\mathbb{E}e^{r\xi^{1-\alpha}} < \infty$  for some*

$$r > r_1 := \frac{2\theta\mathbb{E}\tau_1}{(\text{Var}\xi_1 + v_c^2\text{Var}\tau_1)},$$

*then there exist constants  $r_2, r_3, \dots, r_\gamma \in \mathbb{R}$ , and  $0 < C_1 < C_2 < \infty$  such that*

$$C_1 \exp\left\{-\int_1^x \sum_{j=1}^{\gamma-1} \frac{r_j}{y^{\alpha j}} dy\right\} \leq \psi(x) \leq C_2 \exp\left\{-\int_1^x \sum_{j=1}^{\gamma-1} \frac{r_j}{y^{\alpha j}} dy\right\}. \quad (7.89)$$

Notice that if  $\alpha = 1/(\gamma - 1)$  for a natural  $\gamma \geq 2$ , then

$$\frac{\hat{C}_1}{x^{r_{\gamma-1}}} \exp\left\{-\sum_{j=1}^{\gamma-2} \frac{r_j}{1-\alpha j} x^{1-\alpha j}\right\} \leq \psi(x) \leq \frac{\hat{C}_2}{x^{r_{\gamma-1}}} \exp\left\{-\sum_{j=1}^{\gamma-2} \frac{r_j}{1-\alpha j} x^{1-\alpha j}\right\},$$

and if  $\alpha < 1/(\gamma - 1)$  then

$$\hat{C}_1 \exp\left\{-\sum_{j=1}^{\gamma-1} \frac{r_j}{1-\alpha j} x^{1-\alpha j}\right\} \leq \psi(x) \leq \hat{C}_2 \exp\left\{-\sum_{j=1}^{\gamma-1} \frac{r_j}{1-\alpha j} x^{1-\alpha j}\right\}.$$

*Proof.* We first show that there exist constants  $r_1, r_2, \dots, r_{\gamma-1}$  such that

$$r(x) := \sum_{j=1}^{\gamma-1} \frac{r_j}{x^{\alpha j}}$$

satisfies (2.152). We can determine all these numbers recursively. Indeed, as proven in Proposition 7.14,

$$m_1^{[s(x)]}(x) = \frac{\theta \mathbb{E}\tau_1}{x^\alpha} + o(p_2(x))$$

and

$$m_2^{[s(x)]}(x) = \mathbb{V}ar\xi_1 + v_c^2 \mathbb{V}ar\tau_1 + O(x^{-\alpha})$$

If we now take

$$r_1 = \frac{2\theta \mathbb{E}\tau_1}{\mathbb{V}ar\xi_1 + v_c^2 \mathbb{V}ar\tau_1},$$

then

$$-m_1^{[s(x)]}(x) + \sum_{j=2}^{\gamma} (-1)^j \frac{m_j^{[s(x)]}(x)}{j!} r^{j-1}(x) = O(x^{-2\alpha})$$

for any choice of  $r_2, r_3, \dots, r_{\gamma-1}$ . Then we can choose  $r_2$  such that the coefficient of  $x^{-2\alpha}$  is also zero, and so on.

The conditions (2.155), (2.156) and (2.157) are immediate from the moment assumptions on  $\tau_1$  and  $\xi_1$ . Thus, the bounds on the ruin probability follow from Theorem 2.26.  $\square$

## Chapter 8

# Asymptotically homogeneous Markov chains

In this chapter we are going to consider Markov chains having asymptotically constant (non-zero) drift. As we have seen in the previous chapter, the slower goes  $m_1(x)$  to zero the more moments should have a quite regular behaviour at the infinity. Therefore, it is not surprising that in the case of fixed asymptotical drift one usually considers that the distributions of jumps converge weakly as  $x$  goes to infinity. This corresponds, roughly speaking, to the assumption that *all* moments behave in a regular way. So, we shall say that  $X_n$  is *asymptotically homogeneous in space* if

$$\xi(x) \Rightarrow \xi \quad \text{as } x \rightarrow \infty. \quad (8.1)$$

The class of asymptotically homogeneous chains is a bit larger than the class of additive Markov chains, which has been introduced by Aldous [3], where  $\xi(x)$  are assumed to converge in the total variation norm.

The simplest and one of the most important examples of asymptotically homogeneous Markov chains is a random walk with delay at zero:

$$W_{n+1} = (W_n + \zeta_{n+1})^+, \quad n \geq 0,$$

where  $\{\zeta_k\}$  are independent copies of  $\xi$ . In this example we have also the convergence in total variation. The process  $W_n$  describes the waiting time process in a GI/GI/1 queue.

Another popular class of models closely related to asymptotically homogeneous chains are stochastic recursions

$$Y_{n+1} = A_{n+1}Y_n + B_{n+1}, \quad n \geq 0,$$

where  $\{(A_n, B_n)\}$  are i.i.d. random vectors with values in  $\mathbb{R}^+ \times \mathbb{R}$ . The sequence  $Y_n$  does not satisfy (8.1), but it can be transformed to an asymptotically homogeneous chain, for details see Goldie [30, Section 2].

## 8.1 Key renewal theorem

In this section we shall assume that (8.1) holds and that the mean of the limiting variable  $\xi$  is positive. Our purpose is to study the asymptotic behaviour of the renewal measure.

In contrast to the case of asymptotically zero drift, one can derive a renewal theorem for an asymptotically homogeneous chain  $X_n$  without use of limit theorems for  $X_n$ . Instead, we apply some ideas of the operator approach proposed by Feller [23].

The next theorem is a specialisation of Theorem 1 from Korshunov [45] to the case of transient Markov chains on  $\mathbb{R}^+$ .

**Theorem 8.1.** *Let  $\xi(x) \Rightarrow \xi$  as  $x \rightarrow \infty$  and  $\mathbb{E}\xi > 0$ . Let the family of random variables  $\{|\xi(x)|, x \in \mathbb{R}^+\}$  admit an integrable majorant  $\eta$ , that is,  $\mathbb{E}\eta < \infty$  and*

$$|\xi(x)| \leq_{\text{st}} \eta \quad \text{for all } x \in \mathbb{R}^+. \quad (8.2)$$

*Assume that*

$$\sup_{k \in \mathbb{Z}^+} H(k, k+1] < \infty. \quad (8.3)$$

*If the limit random variable  $\xi$  is non-lattice, then  $H(x, x+h] \rightarrow h/\mathbb{E}\xi$  as  $x \rightarrow \infty$ , for every fixed  $h > 0$ .*

*If the chain  $X_n$  is integer valued and  $\mathbb{Z}$  is the minimal lattice for the variable  $\xi$ , then  $H\{n\} \rightarrow 1/\mathbb{E}\xi$  as  $n \rightarrow \infty$ .*

Condition (8.2) and the dominated convergence theorem imply  $|\xi| \leq_{\text{st}} \eta$ ,  $\mathbb{E}|\xi| < \infty$  and  $\mathbb{E}\xi(x) \rightarrow \mathbb{E}\xi$  as  $x \rightarrow \infty$ ; in particular, the chain  $X_n$  has an asymptotically space-homogeneous drift.

*Proof of Theorem 8.1.* First of all, condition (8.3) allows us to apply Helly's Selection Theorem to the family of measures  $\{H(k + \cdot), k \in \mathbb{Z}^+\}$  (see, for example, Theorem 2 in Section VIII.6 in [23]). Hence, there exists a sequence of points  $t_n \rightarrow \infty$  such that the sequence of measures  $H(t_n + \cdot)$  converges weakly to some measure  $\lambda$  as  $n \rightarrow \infty$ . The following two lemmas characterize  $\lambda$ .

**Lemma 8.2.** *Let  $F$  denote the distribution of  $\xi$ . A weak limit  $\lambda$  of the sequence of measures  $H(t_n + \cdot)$  satisfies the identity  $\lambda = \lambda * F$ .*

*Proof.* The measure  $\lambda$  is non-negative and  $\sigma$ -finite with necessity. Fix any smooth function  $f(x)$  with a bounded support; let  $A > 0$  be such that  $f(x) = 0$  for  $x \notin [-A, A]$ . The weak convergence of measures means the convergence of integrals

$$\int_{-\infty}^{\infty} f(x)H(t_n + dx) = \int_{-A}^A f(x)H(t_n + dx) \rightarrow \int_{-A}^A f(x)\lambda(dx) \quad (8.4)$$

as  $n \rightarrow \infty$ . On the other hand, due to the equality  $H(\cdot) = \mathbb{P}\{X_0 \in \cdot\} + H * P(\cdot)$  we have the following representation for the left side of (8.4):

$$\int_{-A}^A f(x) \mathbb{P}\{X_0 \in t_n + dx\} + \int_{-A}^A f(x) \int_{-\infty}^{\infty} P(t_n + y, t_n + dx) H(t_n + dy). \quad (8.5)$$

Since  $f$  is bounded,

$$\int_{-A}^A f(x) \mathbb{P}\{X_0 \in t_n + dx\} \leq \|f\|_C \mathbb{P}\{X_0 \in [t_n - A, t_n + A]\} \rightarrow 0 \quad (8.6)$$

as  $n \rightarrow \infty$ . The second term in (8.5) is equal to

$$\int_{-\infty}^{\infty} H(t_n + dy) \int_{-A}^A f(x) P(t_n + y, t_n + dx). \quad (8.7)$$

The weak convergence  $P(t, t + \cdot) \Rightarrow F(\cdot)$  as  $t \rightarrow \infty$  implies the convergence of the inner integral in (8.7):

$$\int_{-A}^A f(x) P(t_n + y, t_n + dx) \rightarrow \int_{-A}^A f(x) F(dx - y);$$

here the rate of convergence can be estimated in the following way:

$$\begin{aligned} \Delta(n, y) &:= \left| \int_{-A}^A f(x) (P(t_n + y, t_n + dx) - F(dx - y)) \right| \\ &= \left| \int_{-A}^A f'(x) (\mathbb{P}\{\xi(t_n + y) \leq x - y\} - F(x - y)) dx \right| \\ &\leq \|f'\|_C \int_{-A-y}^{A-y} |\mathbb{P}\{\xi(t_n + y) \leq x\} - F(x)| dx. \end{aligned}$$

Thus, the asymptotic homogeneity of the chain yields for every fixed  $C > 0$  the uniform convergence

$$\sup_{y \in [-C, C]} \Delta(n, y) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (8.8)$$

In addition, by majorisation condition (8.2), for all  $x$  regardless positive or negative,

$$|\mathbb{P}\{\xi(t_n + y) \leq x\} - F(x)| \leq 2\mathbb{P}\{\eta > |x|\}.$$

Hence, for all  $y$ ,

$$\begin{aligned} \Delta(n, y) &\leq 2\|f'\|_C \int_{-A-y}^{A-y} \mathbb{P}\{\eta > |x|\} dx \\ &\leq 4A\|f'\|_C \mathbb{P}\{\eta > |y| - A\}. \end{aligned} \quad (8.9)$$

We have the estimate

$$\begin{aligned}\Delta_n &:= \left| \int_{-\infty}^{\infty} H(t_n + dy) \left( \int_{-\infty}^{\infty} f(x) P(t_n + y, t_n + dx) - \int_{-\infty}^{\infty} f(x) F(dx - y) \right) \right| \\ &\leq \int_{-\infty}^{\infty} \Delta(n, y) H(t_n + dy).\end{aligned}$$

For any fixed  $C > 0$ , uniform convergence (8.8) implies

$$\begin{aligned}\int_{-C}^C \Delta(n, y) H(t_n + dy) &\leq \sup_{y \in [-C, C]} \Delta(n, y) \cdot \sup_n H[t_n - C, t_n + C] \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

The remaining part of the integral can be estimated by (8.9):

$$\begin{aligned}\limsup_{n \rightarrow \infty} \int_{|y| \geq C} \Delta(n, y) H(t_n + dy) \\ \leq 4A \|f'\|_C \limsup_{n \rightarrow \infty} \int_{|y| \geq C} \mathbb{P}\{\eta > |y| - A\} H(t_n + dy).\end{aligned}$$

Since  $\eta$  has finite mean, property (8.3) of the renewal measure  $H$  allows us to choose a sufficiently large  $C$  in order to make the ‘lim sup’ as small as we please. Therefore,  $\Delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, (8.7) has the same limit as the sequence of integrals

$$\int_{-\infty}^{\infty} H(t_n + dy) \int_{-A}^A f(x) F(dx - y).$$

Now the weak convergence to  $\lambda$  implies that (8.7) has the limit

$$\begin{aligned}\int_{-\infty}^{\infty} \lambda(dy) \int_{-\infty}^{\infty} f(x) F(dx - y) &= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} F(dx - y) \lambda(dy) \\ &= \int_{-\infty}^{\infty} f(x) (F * \lambda)(dx).\end{aligned}\tag{8.10}$$

By (8.4)–(8.6) and (8.10), we conclude the identity

$$\int_{-\infty}^{\infty} f(x) \lambda(dx) = \int_{-\infty}^{\infty} f(x) (F * \lambda)(dx).$$

Since this identity holds for every smooth function  $f$  with a bounded support, the measures  $\lambda$  and  $F * \lambda$  coincide. The proof is complete.  $\square$

Further we use the following statement which was proved in Choquet and Deny [14].

**Lemma 8.3.** *Let  $F$  be a distribution not concentrated at 0. Let  $\lambda$  be a nonnegative measure satisfying the equality  $\lambda = F * \lambda$  and the property  $\sup_{n \in \mathbb{Z}} \lambda[n, n + 1] < \infty$ .*

*If  $F$  is non-lattice, then  $\lambda$  is proportional to Lebesgue measure.*

*If  $F$  is lattice with minimal span 1 and  $\lambda(\mathbb{Z}) = 1$ , then  $\lambda$  is proportional to the counting measure.*

The concluding part of the proof of Theorem 8.1 will be carried out for the non-lattice case. Choose any sequence of points  $t_n \rightarrow \infty$  such that the measure  $H(t_n + \cdot)$  converges weakly to some measure  $\lambda$  as  $n \rightarrow \infty$ . It follows from Lemmas 8.2 and 8.3 that then  $\lambda(dx) = \alpha \cdot dx$  with some  $\alpha$ , i.e.,

$$H(t_n + dx) \Rightarrow \alpha \cdot dx \text{ as } n \rightarrow \infty.$$

Now it suffices to prove that  $\alpha = 1/\mathbb{E}\xi$  for all sequences  $t_n$  such that  $H(t_n + \cdot)$  is convergent.

Fix some  $k \in \mathbb{N}$ . Put  $H^{(k)}(\cdot) := U * P^k(\cdot) = \sum_{j=k}^{\infty} \mathbb{P}\{X_j \in \cdot\}$ . Then

$$H^{(k)}(t_n + dx) \Rightarrow \alpha \cdot dx \text{ as } n \rightarrow \infty. \quad (8.11)$$

Consider the measure  $H^{(k)} - H^{(k+1)} = H^{(k)} * (I - P)$ ; by the definition of the renewal measure it is equal to the distribution of  $X_k$ , that is, for any *bounded* Borel set  $B$ ,  $H^{(k)}(B) - H^{(k+1)}(B) = \mathbb{P}\{X_k \in B\}$  (the equality may fail for unbounded sets, say, for  $(x, \infty]$ ). In particular,

$$(H^{(k)} - H^{(k+1)})(0, x] = \mathbb{P}\{X_k \leq x\} \rightarrow 1 \text{ as } x \rightarrow \infty. \quad (8.12)$$

On the other hand,

$$\begin{aligned} & (H^{(k)} - H^{(k+1)})(0, x] \\ &= \int_0^\infty (I - P)(y, (0, x]) H^{(k)}(dy) \\ &= \int_0^x P(y, (x, \infty)) H^{(k)}(dy) - \int_x^\infty P(y, (0, x]) H^{(k)}(dy). \end{aligned} \quad (8.13)$$

The asymptotic homogeneity of the chain and weak convergence (8.11) imply the following convergences of the integrals, for any fixed  $A > 0$ :

$$\int_{t_n-A}^{t_n} P(y, (t_n, \infty)) H^{(k)}(dy) \rightarrow \alpha \int_0^A \mathbb{P}\{\xi > z\} dz \quad (8.14)$$

as  $n \rightarrow \infty$ , and

$$\int_{t_n}^{t_n+A} P(y, (0, t_n]) H^{(k)}(dy) \rightarrow \alpha \int_0^A \mathbb{P}\{\xi \leq -z\} dz. \quad (8.15)$$

Majorisation condition (8.2) allows us to estimate the tails of the integrals:

$$\int_0^{t_n-A} P(y, (t_n, \infty)) H^{(k)}(dy) \leq - \int_A^\infty \mathbb{P}\{\eta > z\} H(t_n - dz) \quad (8.16)$$

and

$$\int_{t_n+A}^\infty P(y, (0, t_n]) H^{(k)}(dy) \leq \int_A^\infty \mathbb{P}\{\eta \geq z\} H(t_n + dz). \quad (8.17)$$

Since the majorant  $\eta$  is integrable, condition (8.3) guarantees that the right sides of inequalities (8.16) and (8.17) can be made as small as we please by the choice of sufficiently large  $A$ . By these reasons we conclude from (8.13)–(8.15) that, as  $n \rightarrow \infty$ ,

$$\begin{aligned} & (H^{(k)} - H^{(k+1)})(0, t_n] \\ & \rightarrow \alpha \int_0^\infty \mathbb{P}\{\xi > z\} dz - \alpha \int_0^\infty \mathbb{P}\{\xi \leq -z\} dz = \alpha \mathbb{E}\xi. \end{aligned}$$

Combining this with (8.12), we infer that  $\alpha = 1/\mathbb{E}\xi$ . Thus, the proof of the theorem is completed.  $\square$

In the next theorem we provide some simple conditions sufficient for condition (8.3), that is, for local compactness of the renewal measure. Denote  $a \wedge b = \min\{a, b\}$ .

**Theorem 8.4.** *Suppose that there exists  $A > 0$  such that*

$$\varepsilon := \inf_{x \in \mathbb{R}^+} \mathbb{E}(\xi(x) \wedge A) > 0. \quad (8.18)$$

*In addition, let*

$$\delta := \inf_{x \in \mathbb{R}^+} \mathbb{P}\{X_n > x \text{ for all } n \geq 1 | X_0 = x\} > 0. \quad (8.19)$$

*Then  $H(x, x+h) \leq (A+h)/\varepsilon\delta$  for all  $x \in \mathbb{R}$  and  $h > 0$ ; in particular, (8.3) holds.*

*Proof.* It suffices to prove that

$$H_y(x, x+h) \leq (A+h)/\varepsilon\delta \quad (8.20)$$

for all  $y \in (x, x+h]$ . Given  $X_0 \in (x, x+h]$ , consider the stopping time

$$T(x+h) = \min\{n \geq 1 : X_n > x+h\}.$$

Since  $X_{T(x+h)} \wedge (x+h+A) - X_0 \leq A+h$  with probability 1,

$$\begin{aligned} A+h & \geq \mathbb{E}(X_{T(x+h)} \wedge (x+h+A) - X_0) \\ & = \sum_{n=1}^{\infty} \mathbb{E}[X_n \wedge (x+h+A) - X_{n-1} \wedge (x+h+A)] \mathbb{I}\{T(x+h) \geq n\}. \end{aligned}$$

Hence, the definition of  $T(x+h)$  implies

$$\begin{aligned} A+h & \geq \sum_{n=1}^{\infty} \mathbb{E}\{X_n \wedge (x+h+A) - X_{n-1} \wedge (x+h+A); T(x+h) \geq n\} \\ & = \sum_{n=1}^{\infty} \mathbb{E}\{X_n \wedge (x+h+A) - X_{n-1} | T(x+h) \geq n\} \mathbb{P}\{T(x+h) \geq n\}. \end{aligned}$$

The Markov property and condition (8.18) yield

$$\mathbb{E}\{X_n \wedge (x+h+A) - X_{n-1} | T(x+h) \geq n\} \geq \mathbb{E}(\xi(X_{n-1}) \wedge A) \geq \varepsilon$$



for all  $n$ . Therefore,

$$A + h \geq \varepsilon \sum_{n=1}^{\infty} \mathbb{P}\{T(x+h) \geq n\} = \varepsilon \mathbb{E}T(x+h).$$

So, the expected number of visits to the interval  $(x, x+h]$  till the first exit from  $(-\infty, x+h]$  does not exceed  $(A+h)/\varepsilon$ , independently of the initial state  $X_0 \in (x, x+h]$ . By condition (8.19), after exit from  $(-\infty, x+h]$  the chain is above the level  $X_T(x+h)$  forever with probability at least  $\delta$ ; in particular, it does not visit the interval  $(x, x+h]$  any more. With probability at most  $1-\delta$  the chain visits this interval again, and so on. Concluding, we get that the expected number of visits to the interval  $(x, x+h]$  cannot exceed the value

$$\frac{A+h}{\varepsilon} \sum_{n=0}^{\infty} (1-\delta)^n = \frac{A+h}{\varepsilon\delta},$$

and (8.20) is proved. The proof of Theorem 8.4 is complete.  $\square$

**Corollary 8.5.** *Let the family of jumps  $\{\xi(x), x \in \mathbb{R}^+\}$  possess an integrable minorant with a positive mean, that is, there exist a random variable  $\zeta$  such that  $\mathbb{E}\zeta > 0$  and  $\xi(x) \geq_{\text{st}} \zeta$  for any  $x \in \mathbb{R}$ . Then*

$$H(x, x+h] \leq (A+h)A/\varepsilon^2$$

for any  $A > 0$  such that  $\varepsilon \equiv \mathbb{E}(\zeta \wedge A) > 0$ ; in particular, (8.3) holds.

*Proof.* Consider the partial sums  $Z_n = \zeta_1 + \dots + \zeta_n$  of an independent copies of  $\zeta$ . Denote the first ascending ladder epoch by  $\chi = \min\{n \geq 1 : Z_n > 0\}$ . It is well known (see, for example, Theorem 2.3(c) in Chapter VIII of [7] that

$$\mathbb{P}\{Z_n > 0 \text{ for all } n \geq 1\} = 1/\mathbb{E}\chi.$$

Since

$$\mathbb{P}\{X_n > x \text{ for all } n \geq 1 | X_0 = x\} \geq \mathbb{P}\{Z_n > 0 \text{ for all } n \geq 1\}$$

by the minorisation condition, the  $\delta$  in Theorem 8.4 is at least  $1/\mathbb{E}\chi$ . Taking into account the inequality  $\mathbb{E}\chi \leq A/\varepsilon$ , we get  $\delta \geq \varepsilon/A$ , which implies the corollary conclusion.

If the chain  $X$  has a non-negative jumps  $\xi(x) \geq 0$ , then the minorisation condition is equivalent to the existence of a positive  $A$  such that

$$\gamma \equiv \inf_{x \in \mathbb{R}} \mathbb{P}\{\xi(x) > A\} > 0. \quad (8.21)$$

In that case one can choose  $\zeta$  taking values 0 and  $A$  with probabilities  $1-\gamma$  and  $\gamma$  respectively; then  $\varepsilon \geq \gamma A$  and  $H(x, x+h] \leq (A+h)/\gamma^2 A$ .  $\square$

In the next theorem we relax a bit the conditions from Theorem 8.4.

**Theorem 8.6.** *Let  $\xi(x) \Rightarrow \xi$  as  $x \rightarrow \infty$  and  $\mathbb{E}\xi > 0$ . Let (8.2) hold. Assume that there exists  $\hat{x} \geq 0$  such that*

$$\varepsilon := \inf_{x \geq \hat{x}} \mathbb{E}(\xi(x) \wedge A) > 0 \quad (8.22)$$

and

$$\delta := \inf_{x \geq \hat{x}} \mathbb{P}_x\{X_n > x \text{ for all } n \geq 1\} > 0. \quad (8.23)$$

*If the limit variable  $\xi$  is non-lattice, then  $H(x, x+h] \rightarrow h/\mathbb{E}\xi$  as  $x \rightarrow \infty$ , for every fixed  $h > 0$ .*

*If the chain  $X_n$  is integer valued and  $\mathbb{Z}$  is the minimal lattice for  $\xi$ , then  $H\{n\} \rightarrow 1/\mathbb{E}\xi$  as  $n \rightarrow \infty$ .*

*Proof.* Set  $h(z) := \mathbb{P}_z\{\min_{k \geq 1} X_k \geq \hat{x}\}$ . Clearly,  $h(z)$  is harmonic for the substochastic kernel  $P(z, dx)\mathbb{I}\{x > \hat{x}\}$ . This means that  $\hat{P}(z, dx) = \frac{h(x)}{h(z)}P(z, dx)\mathbb{I}\{x > \hat{x}\}$  is a stochastic transition kernel. Let  $\hat{X}_n$  denote the corresponding Markov chain. Let  $\hat{\xi}(x)$  denote the jumps of this chain. It is immediate from the definition that

$$\mathbb{P}\{\hat{\xi}(x) \in dy\} = \frac{h(x+y)}{h(x)}\mathbb{P}\{\xi(x) \in dy\}, \quad y > \hat{x} - x.$$

Therefore,  $\hat{\xi}(x) \Rightarrow \xi$  and the family  $\{|\hat{\xi}(x)|\}$  possesses an integrable majorant  $\hat{\eta}$ . Next, noting that (8.23) implies that  $h(x) \geq \delta$  for all  $x \geq \hat{x}$ , we obtain

$$\mathbb{E}(\hat{\xi}(x) \wedge A) \geq \delta \mathbb{E}(\xi(x) \wedge A) \geq \delta \varepsilon$$

and

$$\mathbb{P}_x\{\hat{X}_n \geq x \text{ for all } n \geq 1\} \geq \delta \mathbb{P}_x\{X_n \geq x \text{ for all } n \geq 1\} \geq \delta^2.$$

These two relations imply that  $\hat{X}_n$  satisfies conditions (8.18) and (8.19). Consequently, we may apply Theorem 8.1 to this Markov chain:

$$\hat{H}(x, x+d] \sim \frac{d}{\mathbb{E}\hat{\xi}}, \quad x \rightarrow \infty. \quad (8.24)$$

As usual,  $d$  is an arbitrary positive number in the non-lattice case, and  $d \in \mathbb{N}$  in the lattice case.

It follows from the definition of  $\hat{P}(z, dy)$  that

$$\mathbb{P}_z\{\hat{X}_n \in dy\} = \frac{h(y)}{h(z)}\mathbb{P}_z\left\{X_n \in dy, \min_{1 \leq k \leq n} X_k > \hat{x}\right\}.$$

Therefore,

$$\sum_{n=0}^{\infty} \mathbb{P}_z\left\{X_n \in (x, x+d], \min_{1 \leq k \leq n} X_k > \hat{x}\right\} = h(z) \int_x^{x+d} \frac{1}{h(y)} \hat{H}(dy).$$

Noting that  $h(y) \rightarrow 1$  as  $y \rightarrow \infty$  and using (8.24), we get, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}_z \left\{ X_n \in (x, x+d], \min_{1 \leq k \leq n} X_k > \hat{x} \right\} \\ = h(z) \hat{H}_z(x, x+d] (1 + o(1)) = h(z) \frac{d}{\mathbb{E}\xi} (1 + o(1)). \end{aligned}$$

By the Markov property,

$$\begin{aligned} \min_{y \in [x, x+d]} h(y) \mathbb{P}_z \left\{ X_n \in (x, x+d], \min_{1 \leq k \leq n} X_k > \hat{x} \right\} \\ \leq \mathbb{P}_z \left\{ X_n \in (x, x+d], \min_{k \geq 1} X_k > \hat{x} \right\} \\ \leq \mathbb{P}_z \left\{ X_n \in (x, x+d], \min_{1 \leq k \leq n} X_k > \hat{x} \right\} \end{aligned}$$

Then, as a result,

$$\sum_{n=0}^{\infty} \mathbb{P}_z \left\{ X_n \in (x, x+d], \min_{k \geq 1} X_k > \hat{x} \right\} = h(z) \frac{d}{\mathbb{E}\xi} (1 + o(1)). \quad (8.25)$$

According to Lemma 3.2, the family  $L(x, T(x+d))$  is uniformly integrable. Then, in view of assumption (8.23),  $\sum_{n=0}^{\infty} \mathbb{I}\{X_n \in (x, x+d]\}$  is also uniformly integrable. As a consequence,

$$\lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} \mathbb{P}_z \left\{ X_n \in (x, x+d], \min_{k \geq 1} X_k > \hat{x} \right\} = 0$$

uniformly in  $x$ . This implies that for every  $\gamma > 0$  there exists  $z(\gamma)$  such that

$$(1 - \gamma) \frac{d}{\mathbb{E}\xi} \leq \liminf_{x \rightarrow \infty} H_z(x, x+d] \leq \limsup_{x \rightarrow \infty} H_z(x, x+d] \leq (1 + \gamma) \frac{d}{\mathbb{E}\xi}, \quad z \geq z(\gamma).$$

Since  $\mathbb{P}\{T(z(\gamma)) > N\} \rightarrow 0$  as  $N \rightarrow \infty$  and  $\mathbb{P}\{X_{T(z(\gamma))} > M, T(z(\gamma)) \leq N\} \rightarrow 0$  as  $M \rightarrow \infty$  for each distribution of  $X_0$ , using the uniform integrability of  $\sum_{n=0}^{\infty} \mathbb{I}\{X_n \in (x, x+d]\}$ ,  $x > \hat{x}$ , we get the desired relation.  $\square$

## 8.2 Asymptotics for the stationary distribution

We now turn to the asymptotic behaviour of the stationary distribution of an asymptotically homogeneous chain, that is, we still assume that (8.1) holds. We shall also assume that the limiting variable  $\xi$  satisfies the condition:

$$\text{there exists } \beta > 0 \text{ such that } \mathbb{E}e^{\beta\xi} = 1. \quad (8.26)$$

As is well-known, the stationary measure of the random walk with delay  $W_n$ , say  $\pi_W$ , coincides with the distribution of  $\sup_{n \geq 0} \sum_{k=1}^n \zeta_k$ . Then, by the classical results due to Cramér and Lundberg,

$$\pi_W(x, x+1] \sim ce^{-\beta x} \text{ as } x \rightarrow \infty$$

under the additional assumption  $\mathbb{E}\xi e^{\beta\xi} < \infty$ . Since the jumps of chains  $X_n$  and  $W_n$  are asymptotically equivalent, one can expect that the corresponding stationary distributions have similar asymptotics. This is true on the logarithmic scale only:

**Theorem 8.7.** *Assume that (8.1) and (8.26) are valid. If, additionally,*

$$\sup_{x>0} \mathbb{E} e^{\lambda\xi(x)} < \infty \quad \text{for every } \lambda < \beta \quad (8.27)$$

then

$$\lim_{x \rightarrow \infty} \frac{\log \pi(x, \infty)}{x} = -\beta. \quad (8.28)$$

The lower bound

$$\liminf_{x \rightarrow \infty} \frac{\log \pi(x, \infty)}{x} \geq -\beta$$

is valid for all chains satisfying (8.1) and (8.26), without (8.27). It has been shown in Borovkov and Korshunov [10] via construction of a random walk with delay at zero, which is stochastically smaller than the original chain  $X_n$ .

*Proof.* We derive first a lower bound. To this end we consider the transition kernel

$$Q_\lambda(x, dy) := \frac{e^{\lambda(x-y)}}{\varphi(\lambda)} P(x, dy), \quad y > \hat{x},$$

where  $\lambda \in (0, \beta)$  and  $\varphi(\lambda) := \mathbb{E} e^{\lambda\xi}$ . (An appropriate value of  $\hat{x}$  will be chosen later.) As usual, we set

$$q_\lambda(x) := -\log Q_\lambda(x, \mathbb{R}^+).$$

Let  $X_n^{(\lambda)}$  be a Markov chain embedded in  $Q_\lambda$ , that is, its transition kernel is given by

$$P_\lambda(x, dy) = \frac{Q_\lambda(x, dy)}{Q_\lambda(x, \mathbb{R}^+)}.$$

It follows from the assumptions of the theorem, that

$$\xi^{(\lambda)}(x) \Rightarrow \xi^{(\lambda)},$$

where  $\mathbb{P}\{\xi^{(\lambda)} \in dy\} = \frac{e^{\lambda y}}{\varphi(\lambda)} \mathbb{P}\{\xi \in dy\}$ . Moreover, according to the assumption (8.27), the family  $\xi^{(\lambda)}(x)$  possesses a majorant with exponentially decreasing tails. Choosing  $\lambda$  so that  $\mathbb{E}\xi^{(\lambda)} > 0$  and taking  $\hat{x}$  sufficiently large, we see that the chain  $X_n^{(\lambda)}$  is transient. Furthermore, Theorem 3 in Korshunov [43] implies that  $X_n^{(\lambda)}/n \rightarrow \mathbb{E}\xi^{(\lambda)}$  almost surely.

For the invariant measure  $\pi$  we have the following representation

$$\pi(dy) = e^{-\lambda y} \int_0^{\hat{x}} \pi(dz) e^{\lambda z} \sum_{n=1}^{\infty} \varphi^n(\lambda) \mathbb{E} \left\{ e^{-\sum_{k=0}^{n-1} q_\lambda(X_k^{(\lambda)})}; X_n^{(\lambda)} \in dy \right\}$$

Then, for any positive  $\varepsilon$  any any natural  $m$ ,

$$\begin{aligned} \pi(x, \infty) &> \pi(x, (1 + \varepsilon)x] \\ &> e^{-\lambda(1+\varepsilon)x} \varphi^m(\lambda) \mathbb{E} \left\{ e^{-\sum_{k=0}^{m-1} q_\lambda^+(X_k^{(\lambda)})}; X_m^{(\lambda)} \in (x, (1 + \varepsilon)x) \right\}. \end{aligned} \quad (8.29)$$

It follows from the Jensen inequality that

$$\begin{aligned} &\mathbb{E} \left\{ e^{-\sum_{k=0}^{m-1} q_\lambda^+(X_k^{(\lambda)})} \middle| X_m^{(\lambda)} \in (x, (1 + \varepsilon)x) \right\} \\ &\geq \exp \left\{ -\mathbb{E} \left\{ \sum_{k=0}^{m-1} q_\lambda^+(X_k^{(\lambda)}) \middle| X_m^{(\lambda)} \in (x, (1 + \varepsilon)x) \right\} \right\} \\ &\geq \exp \left\{ -\frac{\mathbb{E} \left\{ \sum_{k=0}^{m-1} q_\lambda^+(X_k^{(\lambda)}) \right\}}{\mathbb{P}\{X_m^{(\lambda)} \in (x, (1 + \varepsilon)x)\}} \right\}. \end{aligned} \quad (8.30)$$

By the law of large numbers for  $X_n^{(\lambda)}$ ,

$$\frac{X_n^{(\lambda)}}{n} \rightarrow \mathbb{E}\xi^{(\lambda)}.$$

Choosing now  $m \sim (1 + \varepsilon/2)x/\mathbb{E}\xi^{(\lambda)}$ , we infer that

$$\mathbb{P}\{X_m^{(\lambda)} \in (x, (1 + \varepsilon)x)\} \rightarrow 1 \quad \text{as } x \rightarrow \infty. \quad (8.31)$$

Combining (8.29), (8.30) and (8.31), we obtain, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \frac{\log \pi(x, \infty)}{x} &\geq -\lambda(1 + \varepsilon) + \frac{1 + \varepsilon/2 + o(1)}{\mathbb{E}\xi^{(\lambda)}} \log \varphi(\lambda) \\ &\quad - (1 + o(1))x^{-1} \mathbb{E} \left\{ \sum_{k=0}^{m-1} q_\lambda^+(X_k^{(\lambda)}) \right\}. \end{aligned} \quad (8.32)$$

By the definition of  $Q_\lambda$ ,

$$Q(x, \mathbb{R}^+) = \frac{1}{\varphi(\lambda)} \mathbb{E} \left\{ e^{\lambda\xi(x)}; \xi(x) > \hat{x} - x \right\} = \frac{\mathbb{E}e^{\lambda\xi(x)}}{\varphi(\lambda)} + O(e^{-\lambda x}) \rightarrow 1.$$

In other words,  $q(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Using the law of large numbers once again, we infer that

$$\mathbb{E} \left\{ \sum_{k=0}^{m-1} q_\lambda^+(X_k^{(\lambda)}) \right\} = o(m) \quad \text{as } m \rightarrow \infty.$$

Applying this to (8.32) and recalling that  $m$  is proportional to  $x$ , we get

$$\liminf_{x \rightarrow \infty} \frac{\log \pi(x, \infty)}{x} \geq -\lambda(1 + \varepsilon) + \frac{1 + \varepsilon/2}{\mathbb{E}\xi^{(\lambda)}} \log \varphi(\lambda).$$

Letting  $\varepsilon \rightarrow 0$ ,  $\lambda \uparrow \beta$  and recaling that  $\varphi(\beta) = 1$ , we finally get

$$\liminf_{x \rightarrow \infty} \frac{\log \pi(x, \infty)}{x} \geq -\beta.$$

The proof of the corresponding upper bound relies on the following equilibrium identity.

**Lemma 8.8.** *Let  $X_n$  be a Markov chain on  $\mathbb{R}^+$ . We shall assume that this chain has invariant distribution  $\pi$ . Let  $V(x) \geq 0$  be a measurable locally bounded functional such that its mean drift  $v(x) := \mathbb{E}\{V(X_1) - V(X_0) | X_0 = x\}$  satisfies the condition*

$$\int v^+(x)\pi(dx) < \infty. \quad (8.33)$$

Then

$$\int v(x)\pi(dx) \geq 0.$$

We postpone the proof of this lemma and complete the proof of the theorem first. Fix some  $\lambda \in (0, \beta)$  and consider  $V_\lambda(x) := e^{\lambda x}$ . Then

$$v_\lambda(x) = \mathbb{E}V_\lambda(x + \xi(x)) - V_\lambda(x) = e^{\lambda x} \left( \mathbb{E}e^{\lambda \xi(x)} - 1 \right).$$

Since  $\mathbb{E}e^{\lambda \xi(x)} \rightarrow \varphi(\lambda) < 1$ , there exists  $\delta > 0$  and  $x_0 = x_0(\delta)$  such that

$$v_\lambda(x) \leq -\delta e^{\lambda x}, \quad x > x_0.$$

Therefore, we may apply Lemma 8.8 to the functional  $V_\lambda$ . As a result, for each  $x \geq x_0$ ,

$$\pi(x, \infty) \leq -\frac{e^{-\lambda x}}{\delta} \int_x^\infty v_\lambda(y)\pi(dy) \leq \frac{e^{-\lambda x}}{\delta} \int_0^{x_0} v_\lambda(y)\pi(dy)$$

Consequently,

$$\limsup_{x \rightarrow \infty} \frac{\log \pi(x, \infty)}{x} \leq -\lambda.$$

Since we may choose  $\lambda$  arbitrary close to  $\beta$ , we arrive at the inequality

$$\limsup_{x \rightarrow \infty} \frac{\log \pi(x, \infty)}{x} \leq -\beta.$$

Thus, the proof of the theorem is complete.  $\square$

*Proof of Lemma 8.8.* Assumption (8.33) implies that  $\int v(x)\pi(dx)$  is defined, but its value can be  $-\infty$ . Assume that  $\int v(x)\pi(dx)$  is negative, i.e.,  $\int v(x)\pi(dx) \leq -3c$  for some positive  $c$ . Then, taking into account (8.33), we conclude that there exists  $A$  such that

$$\int_0^A v(x)\pi(dx) \leq -2c \quad \text{and} \quad \int_A^\infty v^+(x)\pi(dx) < c. \quad (8.34)$$

Let  $N$  be such that  $V(x) \leq N$  for all  $x \leq A$ . Set  $V^*(x) = V(x) \wedge N$ . Then for every  $x \leq A$  we have

$$v^*(x) = \mathbb{E}V^*(x + \xi(x)) - V^*(x) \leq \mathbb{E}V(x + \xi(x)) - V(x) = v(x)$$

Moreover,

$$v^*(x) \leq v^+(x), \quad x \geq 0.$$

From these bounds and from the integrability of  $V^*$  we infer that

$$0 = \int v^*(x)\pi(dx) \leq \int_0^A v(x)\pi(dx) + \int_A^\infty v^+(x)\pi(dx).$$

Combining this with (8.34), we get contradiction to our assumption. Thus, the integral  $\int v(x)\pi(dx)$  is nonnegative.  $\square$

It turns out that the exact (without logarithmic scaling) asymptotic behaviour of  $\pi$  depends not only on the distribution of  $\xi$ , but also on the speed of convergence in (8.1).

Our next result describes the case when the convergence is so fast that the measure  $\pi$  is asymptotically proportional to the stationary measure of  $W_n$ .

**Theorem 8.9.** *Suppose that*

$$\xi(x) \leq_{st} \Xi, \quad x \in \mathbb{R}^+, \quad (8.35)$$

*for some random variable  $\Xi$  such that  $\mathbb{E}\Xi e^{\beta\Xi} < \infty$  and*

$$|\mathbb{E}e^{\beta\xi(x)} - 1|dx \leq q(x) \quad (8.36)$$

*for some decreasing integrable function  $q$ .*

*If the distribution of  $\xi$  is non-lattice then there exists a positive constant  $c$  such that, for every  $d > 0$ ,*

$$\pi(x, x + d] \sim c \left(1 - e^{-\beta d}\right) e^{-\beta x}. \quad (8.37)$$

*If  $X_n$  takes values on  $\mathbb{Z}^+$  and  $\mathbb{Z}$  is the minimal lattice for  $\xi$  then (8.37) holds for all natural  $d$ .*

It is worth mentioning that (8.36) is weaker than conditions we found in the literature. First, Borovkov and Korshunov [10] proved exponential asymptotics for  $\pi$  under the condition

$$\int_0^\infty \int_{-\infty}^\infty e^{\beta y} |\mathbb{P}\{\xi(x) < y\} - \mathbb{P}\{\xi < y\}| dy dx < \infty,$$

which is definitely stronger than (8.36) and implies, in particular, that also the expectations  $\mathbb{E}\xi(x)e^{\beta\xi(x)}$  converge with a summable speed. Furthermore, to show that the constant  $c$  in front of  $e^{-\beta x}$  is positive they introduced the following condition:

$$\int_0^\infty (\mathbb{E}e^{\beta\xi(x)} - 1)^- x \log x dx < \infty.$$

Second, for chains on  $\mathbb{Z}^+$  Foley and McDonald [25] used the assumption, which can be rewritten in our notations as follows

$$\sum_{i=0}^\infty \sum_{j \in \mathbb{Z}} e^{\beta j} |\mathbb{P}\{\xi(i) = j\} - \mathbb{P}\{\xi = j\}| < \infty.$$

Furthermore, the condition (8.36) is quite close to the optimal one. If, for example,  $\mathbb{E}e^{\beta\xi(x)} - 1$  are of the same sign and not summable, then  $\pi(x)e^{\beta x}$  converges either to zero or to infinity, see Corollary 8.13 below. Thus, if (8.36) is violated, then  $\pi(x, x+d]$  may have exponential asymptotics only in the case when  $\mathbb{E}e^{\beta\xi(x)} - 1$  is changing its sign infinitely often.

**Example 8.10.** Consider a Markov chain  $X_n$  on  $\mathbb{Z}^+$  which jumps to the next neighbours only:

$$\mathbb{P}\{\xi(i) = 1\} = 1 - \mathbb{P}\{\xi(i) = -1\} = p + \varphi(i).$$

Assume that, as  $i \rightarrow \infty$ ,

$$\varphi(i) \sim \begin{cases} i^{-\gamma}, & i = 2k \\ -i^{-\gamma}, & i = 2k + 1 \end{cases}$$

with some  $\gamma \in (1/2, 1)$ . Clearly, (8.36) is not satisfied. Let us look at the values of  $X_n$  at even time moments, i.e.,

$$Y_k = X_{2k}, \quad k \geq 0.$$

Then we have

$$\begin{aligned} \mathbb{P}_i\{Y_1 - i = -2\} &= (q - \varphi(i))(q - \varphi(i-1)), \\ \mathbb{P}_i\{Y_1 - i = 0\} &= (q - \varphi(i))(p + \varphi(i-1)) + (p + \varphi(i))(q - \varphi(i+1)), \\ \mathbb{P}_i\{Y_1 - i = 2\} &= (p + \varphi(i))(p + \varphi(i+1)), \end{aligned}$$

where  $q := 1 - p$ . From these equalities we obtain

$$\begin{aligned} \mathbb{E}_i \left[ \left( \frac{q}{p} \right)^{Y_1 - i} \right] - 1 &= \left( \frac{p^2}{q^2} - 1 \right) \mathbb{P}_i\{Y_1 - i = -2\} + \left( \frac{q^2}{p^2} - 1 \right) \mathbb{P}_i\{Y_1 - i = 2\} \\ &= \left( \frac{p^2}{q^2} - 1 \right) (q - \varphi(i))(q - \varphi(i-1)) + \left( \frac{q^2}{p^2} - 1 \right) (p + \varphi(i))(p + \varphi(i+1)) \\ &= -q \left( \frac{p^2}{q^2} - 1 \right) (\varphi(i) + \varphi(i-1)) + p \left( \frac{q^2}{p^2} - 1 \right) (\varphi(i) + \varphi(i+1)) + O(i^{-2\gamma}). \end{aligned}$$

Noting that  $\varphi(i) + \varphi(i+1) = O(i^{-\gamma-1})$ , we conclude that the sequence  $|\mathbb{E}_i(q/p)^{Y_1-i} - 1|$  is summable and, consequently, we may apply Theorem 8.9. Since  $\pi$  is stationary also for  $Y_n$ , we obtain  $\pi(i) \sim c(p/q)^i$  as  $i \rightarrow \infty$ .  $\diamond$

*Proof of Theorem 8.9.* We start, as usual, with the construction of an appropriate Lyapunov function. Let  $p$  be a decreasing regularly varying of index  $-1$  integrable function. Set  $g(x) := \int_x^\infty p(y)dy$  and consider

$$U_p(x) := e^{\beta x}(1 + g(x)).$$

We want to show that there exists  $p(x)$  such that

$$\mathbb{E}U_p(x + \xi(x)) - U_p(x) = -\mathbb{E}\xi^{(\beta)}e^{\beta x}p(x)(1 + o(1)), \quad x \rightarrow \infty. \quad (8.38)$$



By the definition of  $U_p(x)$ ,

$$\begin{aligned} \mathbb{E}U_p(x + \xi(x)) - U_p(x) &= e^{\beta x} \left( \mathbb{E}e^{\beta \xi(x)}(1 + g(x + \xi(x))) - 1 - g(x) \right) \\ &= e^{\beta x}(1 + g(x)) \left( \mathbb{E}e^{\beta \xi(x)} - 1 \right) + e^{\beta x} \mathbb{E}(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}. \end{aligned} \quad (8.39)$$

The assumption (8.36) yields the existence of  $p(x)$  such that

$$|\mathbb{E}e^{\beta \xi(x)} - 1| = o(p(x)). \quad (8.40)$$

Fix some increasing function  $s(x) = o(x)$  and split the second term in (8.39) into three parts:

$$\begin{aligned} \mathbb{E}(g(x + \xi(x)) - g(x))e^{\beta \xi(x)} &= \mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; \xi(x) < -s(x)\} \\ &\quad + \mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; \xi(x) > s(x)\} \end{aligned}$$

Due to the monotonicity of  $g$ ,

$$0 \leq \mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; \xi(x) < -s(x)\} \leq g(0)e^{-\beta s(x)}. \quad (8.41)$$

Since  $p(x)$  is regularly varying,  $g(x + \xi(x)) - g(x) \sim -p(x)\xi(x)$  uniformly on the set  $|\xi(x)| \leq s(x)$ . Therefore,

$$\mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; |\xi(x)| \leq s(x)\} \sim -p(x)\mathbb{E}\{\xi(x)e^{\beta \xi(x)}; |\xi(x)| \leq s(x)\}.$$

Recalling that the family  $\xi(x)$  possesses a majorant  $\Xi$  with  $\mathbb{E}\Xi e^{\beta \Xi} < \infty$ , we infer that  $\mathbb{E}\{\xi(x)e^{\beta \xi(x)}; |\xi(x)| \leq s(x)\} \rightarrow \mathbb{E}\xi^{(\beta)}$ . As a result,

$$\mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; |\xi(x)| \leq s(x)\} \sim -p(x)\mathbb{E}\xi^{(\beta)}. \quad (8.42)$$

The existence of  $\Xi$  implies also that the function  $\mathbb{E}\{e^{\beta \xi(x)}; \xi(x) > s(x)\}$  is bounded by  $\mathbb{E}\{e^{\beta \Xi}; \Xi > s(x)\}$ . The latter function is decreasing and summable provided that  $s(x)/x \rightarrow 0$  sufficiently slow. Consequently, there exists  $p(x)$  such that

$$\mathbb{E}\{(g(x + \xi(x)) - g(x))e^{\beta \xi(x)}; \xi(x) > s(x)\} = o(p(x)). \quad (8.43)$$

Combining (8.41)–(8.43), we conclude that

$$\mathbb{E}(g(x + \xi(x)) - g(x))e^{\beta \xi(x)} = -p(x)\mathbb{E}\xi^{(\beta)}(1 + o(1)).$$

Plugging this relation and (8.40) into (8.39), we obtain (8.38).

We next determine the asymptotic behaviour of  $\mathbb{E}\xi(x)U_p(x + \xi(x))$ . We start by splitting this expectation into three parts:

$$\begin{aligned} \mathbb{E}\xi(x)U_p(x + \xi(x)) &= \mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) < -s(x)\} \\ &\quad + \mathbb{E}\{\xi(x)U_p(x + \xi(x)); |\xi(x)| \leq s(x)\} \\ &\quad + \mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) > s(x)\} \end{aligned}$$

It follows from the definition of  $U_p$  that

$$\mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) < -s(x)\} \geq (1 + g(0))e^{-\beta s(x)}U_p(x)\mathbb{E}\{\xi(x); \xi(x) < -s(x)\}$$

and

$$\mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) > s(x)\} \leq U_p(x)\mathbb{E}\{\xi(x)e^{\beta \xi(x)}; \xi(x) > s(x)\}.$$

Then, the integrability of the majorant  $\Xi$  implies that these two expectations are  $o(U_p(x))$ . Finally,

$$\begin{aligned} \mathbb{E}\{\xi(x)U_p(x + \xi(x)); |\xi(x)| \leq s(x)\} &\sim U_p(x)\mathbb{E}\{\xi(x)e^{\beta \xi(x)}; |\xi(x)| \leq s(x)\} \\ &\sim U_p(x)\mathbb{E}\xi^{(\beta)}. \end{aligned}$$

As a result,

$$\mathbb{E}\xi(x)U_p(x + \xi(x)) = (1 + o(1))\mathbb{E}\xi^{(\beta)}U_p(x), \quad x \rightarrow \infty. \quad (8.44)$$

Consider, as usual, the transition kernel

$$Q(x, dy) = \frac{U_p(y)}{U_p(x)}P(x, dy), \quad y \geq \hat{x}.$$

It follows from (8.38) that, for all  $\hat{x}$  sufficiently large,

$$\begin{aligned} Q(x, \mathbb{R}^+) &= \frac{1}{U_p(x)}\mathbb{E}\{U_p(x + \xi(x)); \xi(x) \geq \hat{x} - x\} \\ &\leq \frac{1}{U_p(x)}\mathbb{E}\{U_p(x + \xi(x))\} \leq 1, \quad x \geq \hat{x}. \end{aligned} \quad (8.45)$$

In other words,  $Q$  is substochastic. Furthermore, it follows from the asymptotic homogeneity that

$$Q(x, \mathbb{R}^+) \geq \mathbb{P}\{\xi(x) \geq 0\} \geq \mathbb{P}\{\xi \geq 0\}/2, \quad x \geq \hat{x}. \quad (8.46)$$

Using (8.38) once again, we conclude that

$$q(x) = -\log Q(x, \mathbb{R}^+) = O(p(x)), \quad x \rightarrow \infty. \quad (8.47)$$

Let  $\hat{X}_n$  be a Markov chain with the transition kernel

$$\hat{P}(x, dy) = \frac{Q(x, dy)}{Q(x, \mathbb{R}^+)}$$

and let  $\hat{\xi}(x)$  denote its jumps. Then, it follows from (8.44) and (8.45) that, for all  $\hat{x}$  sufficiently large,

$$\begin{aligned} \mathbb{E}\hat{\xi}(x) &= \frac{\mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) \geq \hat{x} - x\}}{U_p(x)Q(x, \mathbb{R}^+)} \\ &\geq \frac{\mathbb{E}\{\xi(x)U_p(x + \xi(x))\}}{U_p(x)} \geq \frac{\mathbb{E}\xi^{(\beta)}}{2}, \quad x \geq \hat{x}. \end{aligned} \quad (8.48)$$

It is immediate from the definition of  $U_p$  that  $\hat{\xi}(x) \Rightarrow \xi^{(\beta)}$ . Furthermore, the assumption that  $\mathbb{E}\Xi e^{\beta\Xi} < \infty$  and (8.46) imply that the family  $\hat{\xi}(x)$  possesses an integrable majorant. From the latter we infer that (8.22) is valid for all  $A$  sufficiently large. In order to show (8.23) we will construct a stochastic minorant for  $\{\xi(x); x > \hat{x}\}$ , which has positive expectation. But the existence of such a minorant is immediate from the weak convergence  $\hat{\xi}(x) \Rightarrow \xi^{(\beta)}$  and from the fact that the family  $e^{\theta\xi^-(x)}$  is uniformly integrable for each  $\theta \in (0, \beta)$ . Thus, we may apply Theorem 8.6 to the chain  $\hat{X}_n$ : If  $\xi$  is non-lattice then, for every  $h > 0$ ,

$$\hat{H}(x, x+h] \sim \frac{h}{\mathbb{E}\xi^{(\beta)}}.$$

If  $\mathbb{Z}$  is the minimal lattice for  $\xi$  then the previous relation is valid for every natural  $h$ . Combining (8.47) with the bound  $\hat{H}(x, x+h] \leq C$  we conclude that  $\sum_{k=0}^{\infty} \mathbb{E}q(\hat{X}_k) < \infty$ . Thus, by Lemma 3.4,

$$\hat{H}^{(q)}(x, x+h] \sim \frac{h}{\mathbb{E}\xi^{(\beta)}} \mathbb{E}e^{-\sum_{k=0}^{\infty} q(\hat{X}_k)}. \quad (8.49)$$

Here, again,  $h$  is arbitrary positive number in the case when  $\xi$  is non-lattice and  $h$  is natural in the lattice case.

For the invariant distribution  $\pi$  we have the representation.

$$\pi(dy) = \frac{\hat{H}^{(q)}(dy)}{U_p(y)} \int_0^{\hat{x}} U_p(z) \pi(dz)$$

If  $\xi$  is lattice then

$$\pi\{n\} = \frac{\hat{H}^{(q)}\{n\}}{U_p(n)} \int_0^{\hat{x}} U_p(z) \pi(dz).$$

The result follows now from (8.49) and the fact that  $U_p(x) \sim e^{\beta x}$ .

Consider now the non-lattice case. For every fixed  $h > 0$ ,

$$\frac{\hat{H}^{(q)}(x, x+h]}{\max_{x \leq y \leq x+h} U_p(y)} \int_0^{\hat{x}} U_p(z) \pi(dz) \leq \pi(x, x+h] \leq \frac{\hat{H}^{(q)}(x, x+h]}{\min_{x \leq y \leq x+h} U_p(y)} \int_0^{\hat{x}} U_p(z) \pi(dz).$$

Using again (8.49), we obtain the bounds

$$che^{-\beta x - \beta h}(1 + o(1)) \leq \pi(x, x+h] \leq che^{-\beta x}(1 + o(1)).$$

Taking  $h = d/m$ , we then get

$$ce^{-\beta x - d\beta/m}(1 + o(1)) \sum_{k=0}^{m-1} \frac{1}{m} e^{-\beta dk/m} \leq \pi(x, x+d] \leq ce^{-\beta x}(1 + o(1)) \sum_{k=0}^{m-1} \frac{1}{m} e^{-\beta dk/m}.$$

Since we may choose  $m$  arbitrary large, we obtain

$$\pi(x, x+d] \sim c(1 - e^{-\beta d})e^{-\beta x}.$$

Thus, the proof of the theorem is complete.  $\square$

We now turn to the case where  $\mathbb{E}e^{\beta\xi(x)}$  converges to 1 in a non-summable way. Our next result describes the behaviour of  $\pi$  in terms of a non-uniform exponential change of measure.

**Theorem 8.11.** *Suppose that, for some  $\varepsilon > 0$ ,*

$$\sup_{x \in \mathbb{R}^+} \mathbb{E}e^{(\beta+\varepsilon)\xi(x)} < \infty. \quad (8.50)$$

*Assume also that there exists a differentiable function  $\beta(x)$  such that*

$$|\mathbb{E}e^{\beta(x)\xi(x)} - 1|dx \leq \gamma(x), \quad (8.51)$$

*and  $|\beta'(x)| \leq \gamma(x)$  where  $\gamma(x)$  is a decreasing integrable function. Then, for some  $c > 0$ ,*

$$\pi(x, x+d] \sim c(1 - e^{-\beta d})e^{-\int_0^x \beta(y)dy} \quad \text{as } x \rightarrow \infty,$$

*where  $d$  is an arbitrary positive number in the non-lattice case and natural in the lattice case.*

*Proof of Theorem 8.11.* The proof is quite similar to the proof of Theorem 8.9, the only difference consists in the Lyapunov function  $U_p$ . Instead of  $(1 + g(x))e^{\beta x}$  we shall use

$$U_p(x) := (1 + g(x))e^{\int_0^x \beta_\varepsilon(y)dy},$$

where  $\beta_\varepsilon(y) := \max\{\beta(y), \varepsilon/2\}$ .

Observe that, with necessity,  $\beta(x) \rightarrow \beta$  so that, by the condition (8.50),

$$\mathbb{E}\left\{e^{\int_x^{x+\xi(x)} \beta_\varepsilon(y)dy}; |\xi(x)| > \sqrt{x}\right\} = o(e^{-\varepsilon\sqrt{x}/2}) \quad \text{as } x \rightarrow \infty.$$

Convergence  $\beta(x) \rightarrow \beta$  implies also that  $\beta_\varepsilon(x) = \beta(x)$  for all sufficiently large values of  $x$ . Further, condition on the derivative of  $\beta(y)$  implies that

$$\begin{aligned} \left| \int_x^{x+\xi(x)} \beta(y)dy - \beta(x)\xi(x) \right| &\leq \int_x^{x+\xi(x)} |\beta(y) - \beta(x)|dy \\ &\leq \sup_{|y| \leq \sqrt{x}} |\beta'(x+y)|\xi^2(x)/2 \\ &\leq \gamma(x - \sqrt{x})\xi^2(x)/2. \end{aligned}$$

Uniformly in  $|\xi(x)| \leq \sqrt{x}$ , we have  $\gamma(x - \sqrt{x})\xi^2(x) \leq \gamma(x - \sqrt{x})x \rightarrow 0$  as  $x \rightarrow \infty$ . Therefore, again in view of the condition (8.50),

$$\mathbb{E}\left\{e^{\int_x^{x+\xi(x)} \beta_\varepsilon(y)dy}; |\xi(x)| \leq \sqrt{x}\right\} = \mathbb{E}e^{\beta(x)\xi(x)} + O(\gamma(x - \sqrt{x}) + e^{-\varepsilon\sqrt{x}/2}), \quad x \rightarrow \infty.$$

Hence,

$$\mathbb{E}\left\{e^{\int_x^{x+\xi(x)} \beta_\varepsilon(y)dy}\right\} = \mathbb{E}e^{\beta(x)\xi(x)} + O(\gamma(x - \sqrt{x}) + e^{-\varepsilon\sqrt{x}/2}), \quad (8.52)$$

Taking into account (8.51), we conclude that there exists a decreasing integrable function  $p_1(x)$  such that

$$\mathbb{E} \left\{ e^{\int_x^{x+\xi(x)} \beta_\varepsilon(y) dy} \right\} = 1 + O(p_1(x)). \quad (8.53)$$

Similar to (8.39), we have

$$\begin{aligned} \mathbb{E}U_p(x + \xi(x)) - U_p(x) &= U_p(x) \left( \mathbb{E}e^{\int_x^{x+\xi(x)} \beta_\varepsilon(y) dy} - 1 \right) \\ &\quad + e^{\int_0^x \beta_\varepsilon(y) dy} \mathbb{E}(g(x + \xi(x)) - g(x)) e^{\int_x^{x+\xi(x)} \beta_\varepsilon(y) dy}. \end{aligned}$$

Using (8.52) and (8.53), and recalling that  $g(x)$  is bounded, we get

$$\begin{aligned} \mathbb{E}U_p(x + \xi(x)) - U_p(x) &= O(p_1(x)U_p(x)) \\ &\quad + e^{\int_0^x \beta_\varepsilon(y) dy} \mathbb{E}(g(x + \xi(x)) - g(x)) e^{\beta(x)\xi(x)}. \end{aligned}$$

Repeating the corresponding arguments in the proof of Theorem 8.9 and using (8.50), we obtain

$$\mathbb{E} \left\{ (g(x + \xi(x)) - g(x)) e^{\beta(x)\xi(x)}; |\xi(x)| > \sqrt{x} \right\} = o(e^{-\varepsilon\sqrt{x}/2})$$

and

$$\mathbb{E} \left\{ (g(x + \xi(x)) - g(x)) e^{\beta(x)\xi(x)}; |\xi(x)| \leq \sqrt{x} \right\} \sim -p(x) \mathbb{E} \left\{ \xi(x) e^{\beta(x)\xi(x)} \right\}.$$

Therefore, taking  $p(x) \gg p_1(x)$ , we get

$$\mathbb{E}U_p(x + \xi(x)) - U_p(x) \sim -p(x)U_p(x) \mathbb{E} \left\{ \xi(x) e^{\beta(x)\xi(x)} \right\}.$$

Using (8.50) once again, we have  $\mathbb{E} \left\{ \xi(x) e^{\beta(x)\xi(x)} \right\} \rightarrow \mathbb{E}\xi^{(\beta)}$ . Consequently,

$$\mathbb{E}U_p(x + \xi(x)) - U_p(x) = -p(x)U_p(x) \mathbb{E}\xi^{(\beta)}(1 + o(1)), \quad x \rightarrow \infty. \quad (8.54)$$

Repeating the proof of (8.44) with the new function  $U_p$ , one can easily see that this relation is still valid. This means that  $U_p$  is an appropriate Lyapunov function, and the remaining part of the proof is a word for word repetition of the proof of Theorem 8.9.  $\square$

Since  $\beta(x)$  is not given explicitly, Theorem 8.11 can not be seen as a final statement. For this reason we describe below some situation where  $\beta(x)$  can be expressed via the difference  $\mathbb{E}e^{\beta\xi(x)} - 1$ .

**Corollary 8.12.** *Assume the condition (8.50) and that there exists a differentiable function  $\alpha(x)$  such that  $\alpha'(x)$  is regularly varying at infinity with index  $-2 < r < -3/2$  and*

$$\mathbb{E}e^{\beta\xi(x)} - 1 = \alpha(x) + O(\gamma(x)), \quad (8.55)$$

where  $\gamma(x)$  a decreasing integrable function. Suppose also that

$$\alpha(x)(\mathbb{E}\xi(x)e^{\beta\xi(x)} - m) = O(\gamma(x)), \quad (8.56)$$

where  $m := \mathbb{E}\xi e^{\beta\xi}$ . Then

$$\pi(x, x+d] \sim ce^{-\beta x + A(x)/m} \quad \text{as } x \rightarrow \infty, \quad (8.57)$$

where  $c > 0$  and  $A(x) := \int_0^x \alpha(y)dy$ .

*Proof.* Notice that, since  $-2 < r < -3/2$ ,  $A(x) \rightarrow \infty$ ,  $A(x) = o(x)$  as  $x \rightarrow \infty$  and  $\int_1^\infty \alpha^2(x)dx < \infty$ .

Take  $\beta(x) := \beta - \alpha(x)/m$ . Since  $r < -3/2$ ,  $\alpha(x) = o(1/\sqrt{x})$ . Hence, by Taylor's theorem, uniformly in  $|\xi(x)| \leq \sqrt{x}$ ,

$$e^{-\alpha(x)\xi(x)/m} = 1 - \alpha(x)\xi(x)/m + O(\alpha^2(x)\xi^2(x)).$$

which yields

$$\begin{aligned} \mathbb{E}e^{\beta(x)\xi(x)} &= \mathbb{E}e^{\beta\xi(x)} - \alpha(x)\mathbb{E}\xi(x)e^{\beta\xi(x)}/m + O(\alpha^2(x)) \\ &= \mathbb{E}e^{\beta\xi(x)} - \alpha(x) + O(|\alpha(x)(\mathbb{E}\xi(x)e^{\beta\xi(x)} - m)| + \alpha^2(x)) \\ &= 1 + O(\gamma(x) + |\alpha(x)(\mathbb{E}\xi(x)e^{\beta\xi(x)} - m)| + \alpha^2(x)). \end{aligned}$$

Thus, the function  $\beta(x)$  satisfies all the conditions of Theorem 8.11 and the proof is complete.  $\square$

Notice that the key condition on the rate of convergence of  $\mathbb{E}e^{\beta\xi(x)}$  to 1 that implies asymptotics (8.57) in the latter corollary is that the sequence  $\alpha^2(x)$  is summable. If it is not so, that is, if the index  $r+1$  of regular variation of the function  $\alpha(x)$  is between  $-1/2$  and 0, then the asymptotic behaviour of  $\pi(x, x+1]$  is different from (8.57) which is specified in the following corollary.

**Corollary 8.13.** *Assume the condition (8.50) and that there exists a differentiable function  $\alpha(x)$  such that*

$$|\alpha(x)| \leq \frac{c}{(1+x)^{\frac{1}{M+1}+\varepsilon}}$$

for some  $c < \infty$ ,  $M \in \mathbb{N}$ , and  $\varepsilon > 0$ ,

$$|\alpha'(x)| \leq \gamma(x) \quad (8.58)$$

and

$$\mathbb{E}e^{\beta\xi(x)} - 1 = \alpha(x) + O(\gamma(x))$$

for some decreasing integrable  $\gamma(x)$ . Assume also that, for every  $k = 1, 2, \dots, M$ ,

$$m_k(x) = m_k + \sum_{j=1}^{M-k} D_{k,j}\alpha^j(x) + O(\alpha^{M-k+1}(x)), \quad (8.59)$$

where  $m_k(x) := \mathbb{E}\xi^k(x)e^{\beta\xi(x)}$  and  $m_k := \mathbb{E}\xi^k e^{\beta\xi}$ . Then there exist real numbers  $R_1, R_2, \dots, R_M$  such that

$$\pi(x, x+d] \sim c \exp\left\{-\beta x - \sum_{k=1}^M R_k \int_0^x \alpha^k(y) dy\right\} \quad \text{as } x \rightarrow \infty. \quad (8.60)$$

*Proof.* Define

$$\Delta(x) := \sum_{k=1}^M R_k \alpha^k(x).$$

In view of Theorem 8.11 it suffices to show that there exist  $R_1, R_2, \dots, R_M$  such that

$$\left| \mathbb{E} e^{(\beta + \Delta(x))\xi(x)} - 1 \right| \leq q(x) \quad (8.61)$$

for some decreasing integrable function  $q(x)$ . Indeed,  $\Delta(x)$  is differentiable and  $|\Delta'(x)| \leq C|\alpha'(x)|$ . Therefore, we may apply Theorem 8.11 with  $\beta(x) = \beta + \Delta(x)$ .

By Taylor's theorem, the calculations similar to the previous corollary show that, as  $i \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{E} e^{(\beta + \Delta(x))\xi(x)} &= \mathbb{E} e^{\beta\xi(x)} + \sum_{k=1}^M \frac{m_k(x)}{k!} \Delta^k(x) + O(\Delta^{M+1}(x)) \\ &= 1 + \alpha(x) + \sum_{k=1}^M \frac{m_k(x)}{k!} \Delta^k(x) + O(\gamma(x) + \alpha^{M+1}(x) + e^{-\varepsilon x/2}). \end{aligned}$$

From this equality we infer that we may determine  $R_1, R_2, \dots, R_M$  by the relation

$$\alpha(x) + \sum_{k=1}^M \frac{m_k(x)}{k!} \Delta^k(x) = O(\alpha^{M+1}(x)). \quad (8.62)$$

It follows from the assumption (8.59) and the bound  $\Delta(x) = O(\alpha(x))$  that (8.62) is equivalent to

$$z + \sum_{k=1}^M \frac{1}{k!} \left( m_k + \sum_{j=1}^{M-k} D_{k,j} z^j \right) \left( \sum_{j=1}^M R_j z^j \right)^k = O(z^{M+1}) \quad \text{as } z \rightarrow 0.$$

Consequently, the coefficient at  $z^k$  should be zero for every  $k \leq M$ , and we can determine all  $R_k$  recursively. For example, the coefficient at  $z$  equals  $1 + m_1 R_1$ . Thus,  $R_1 = -1/m_1$ . Further, the coefficient at  $z^2$  is  $D_{1,1} R_1 + m_1 R_2 + m_2 R_1^2/2$  and, consequently,

$$R_2 = \frac{D_{1,1}}{m_1^2} - \frac{m_2}{2m_1^3}.$$

All further coefficients can be found in the same way.  $\square$

If  $\alpha(x)$  from Corollary 8.13 decreases slower than any power of  $x$  but (8.58) and (8.59) remain valid, then one has, by the same arguments,

$$\pi(x, x+1] = \exp \left\{ -\beta x - \sum_{k=1}^M R_k \int_0^x \alpha^k(y) dy + O \left( \int_0^x \alpha^{M+1}(y) dy \right) \right\}$$

which can be seen as a corrected logarithmic asymptotic for  $\pi$ . To obtain precise asymptotics one needs more information on the moments  $m_k(x)$ .

**Corollary 8.14.** *Assume the condition (8.50) and that there exists a differentiable function  $\alpha(x)$  such that (8.58) holds,*

$$\mathbb{E} e^{\beta \xi(x)} - 1 = \alpha(x), \quad x \geq 0 \quad (8.63)$$

and

$$m_k(x) = m_k + \sum_{j=1}^{\infty} D_{k,j} \alpha^j(x) \quad (8.64)$$

for all  $k \geq 1$ . Assume furthermore that  $\sup_{k \geq 1} \sum_{j=1}^{\infty} D_{k,j} r^j < \infty$  for some  $r > 0$ . Then there exist real numbers  $R_1, R_2, \dots$ ,

$$\pi(x, x+d] \sim c \exp \left\{ -\beta x - \sum_{k=1}^{\infty} R_k \int_0^x \alpha^k(y) dy \right\}.$$

*Proof.* For every  $x \geq 0$  let  $\beta(x)$  denote the positive solution of the equation

$$\mathbb{E} e^{\beta(x) \xi(x)} = 1.$$

Since  $\mathbb{E} e^{\beta \xi(x)}$  is finite for all  $\gamma \leq \beta + \varepsilon$ , we may rewrite the latter equation as Taylor's series:

$$\mathbb{E} e^{\beta \xi(x)} + \sum_{k=1}^{\infty} \frac{\Delta^k(x)}{k!} \mathbb{E} \xi^k(x) e^{\beta \xi(x)} = 1,$$

where  $\Delta(x) = \beta(x) - \beta$ . Taking into account (8.63) and (8.64), we then get

$$\alpha(x) + \sum_{k=1}^{\infty} \frac{\Delta^k(x)}{k!} \sum_{j=0}^{\infty} D_{k,j} \alpha^j(x) = 0, \quad x \geq 0. \quad (8.65)$$

Set  $D_{0,1} = 1$  and define

$$F(z, w) := \sum_{k,j \geq 0} \frac{D_{k,j}}{k!} z^j w^k.$$

Therefore, (8.65) can be written as  $F(\alpha(x), \Delta(x)) = 0$ . In other words, we are looking for a function  $w(z)$  satisfying  $F(z, w(z)) = 0$ . Since  $F(0, 0) = 0$  and  $\frac{\partial}{\partial w} F(0, 0) = m_1 > 0$ , we may apply Theorem B.4 from Flajolet and Sedgewick [24] which says that  $w(z)$  is analytic in a vicinity of zero, that is, there exists  $\rho > 0$  such that

$$w(z) = \sum_{n=1}^{\infty} R_n z^n, \quad |z| < \rho.$$



Consequently,

$$\Delta(x) = \sum_{n=1}^{\infty} R_n \alpha^n(x)$$

for all  $x$  such that  $|\alpha(x)| < \rho$ .

Applying Theorem 8.11 with  $\beta(x) = \beta + \Delta(x)$ , we get

$$\pi(x, x + d] \sim ce^{-\beta x - \int_0^x \Delta(y) dy}.$$

Integrating  $\Delta(y)$  piecewise, we complete the proof.  $\square$

We finish with the following remark. In the proof of Corollary 8.14 we have adapted the derivation of the Cramér series in large deviations for sums of independent random variables, see, for example, Petrov [60]. There is just one difference: we needed analyticity of an implicit function instead of analyticity of an inverse function.

### 8.3 Local central limit theorem

We first state a version of the central limit theorem for Markov chains with asymptotically constant drift.

**Theorem 8.15.** *Let the family  $\{\xi(x)\}$  possess a square integrable majorant. If  $m_1(x) = \mu + o(1/\sqrt{x})$  and  $m_2(x) \rightarrow b > 0$  then*

$$\frac{X_n - \mu n}{\sqrt{bn}} \Rightarrow N_{0,1}$$

and

$$\frac{\max_{k \leq n} X_k - \mu n}{\sqrt{bn}} \Rightarrow N_{0,1}.$$

These statements are easy consequences of Theorem 3.16 and Theorem 3.17 respectively, and we omit their proofs.

**Theorem 8.16.** *Let the family  $\{\xi(x)\}$  possess a stochastic minorant with positive mean and a square integrable stochastic majorant. Assume that  $\xi(x) \Rightarrow \xi$  and that  $m_1(x) = \mu + o(1/\sqrt{x})$ .*

*If  $\xi$  is non-lattice and, for every  $A > 0$ ,*

$$\sup_{|\lambda| \leq A} \left| \mathbb{E} e^{i\lambda \xi(x)} - \mathbb{E} e^{i\lambda \xi} \right| = o(1/x), \quad (8.66)$$

*then, for each  $h > 0$ ,*

$$\sup_{x \in \mathbb{R}^+} \left| \sqrt{2\pi bn} \mathbb{P}\{X_n \in (x, x + h]\} - h e^{-(x - n\mu)^2 / 2bn} \right| = o(1).$$

*If  $\xi$  is integer valued,  $\mathbb{Z}$  is the minimal lattice for  $\xi$  and*

$$\sup_{|\lambda| \leq \pi} \left| \mathbb{E} e^{i\lambda \xi(x)} - \mathbb{E} e^{i\lambda \xi} \right| = o(1/x), \quad (8.67)$$

then

$$\sup_{x \in \mathbb{Z}^+} \left| \sqrt{2\pi bn} \mathbb{P}\{X_n = x\} - e^{-(x-n\mu)^2/2bn} \right| = o(1).$$

*Proof.* We give a proof for the lattice case only, the non-lattice case can be treated similarly. By the inversion formula,

$$\sqrt{n} \mathbb{P}\{X_n = x\} = \frac{1}{2\pi} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-i\lambda \frac{x-n\mu}{\sqrt{n}}} \mathbb{E} e^{i\lambda \frac{X_n-n\mu}{\sqrt{n}}} d\lambda.$$

Therefore, using standard arguments,

$$\begin{aligned} \sup_x \left| \sqrt{n} \mathbb{P}\{X_n = x\} - \frac{1}{\sqrt{2\pi b}} e^{-(x-n\mu)^2/2b} \right| &\leq \frac{1}{2\pi} \int_{-A}^A \left| \mathbb{E} e^{i\lambda \frac{X_n-n\mu}{\sqrt{n}}} - e^{-\lambda^2 b/2} \right| d\lambda \\ &+ \int_{|\lambda| \in (A, \pi\sqrt{n}]} \left| \mathbb{E} e^{i\lambda \frac{X_n-n\mu}{\sqrt{n}}} \right| d\lambda + \int_{|\lambda| > A} e^{-\lambda^2 b/2} d\lambda. \end{aligned} \quad (8.68)$$

It follows from the weak convergence  $\frac{X_n-n\mu}{\sqrt{n}} \Rightarrow N_{0,b}$  that  $\mathbb{E} e^{i\lambda \frac{X_n-n\mu}{\sqrt{n}}} \rightarrow e^{-\lambda^2 b/2}$  uniformly on compact sets. Therefore, the first integral on the right hand side of (8.68) converges to zero for each fixed  $A$ . Choosing  $A$  large we can make the integral  $\int_{|\lambda| > A} e^{-\lambda^2 b/2} d\lambda$  arbitrary small. Thus, the proof will be completed if we show that the second integral in (8.68) also converges to zero.

Let us introduce an auxiliary Markov chain  $\tilde{X}_k$ . Define

$$\theta_n := \inf\{k > n/2 : X_k \leq n\mathbb{E}\eta/4\},$$

where  $\eta$  is a minorant for the family  $\{\xi(x)\}$ . For every  $k \leq \theta_n$  we set  $\tilde{X}_k = X_k$ . After the stopping time  $\theta_n$  we let  $\tilde{X}_k$  have the jumps

$$\tilde{\xi}(x) = \begin{cases} \xi(x), & x > n\mathbb{E}\eta/4, \\ \xi(x_n), & x \leq n\mathbb{E}\eta/4, \end{cases}$$

where  $x_n > n\mathbb{E}\eta/4$  is a fixed integer. It is clear from the construction that

$$|\mathbb{E} e^{iuX_n}| \leq |\mathbb{E} e^{iu\tilde{X}_n}| + \mathbb{P}\{\theta_n < n\}, \quad u \in \mathbb{R}. \quad (8.69)$$

Let  $\{\eta_k\}$  be independent copies of the minorant  $\eta$  and set  $S_k := \eta_1 + \eta_2 + \dots + \eta_k$ . Then

$$\begin{aligned} \mathbb{P}\{\theta_n < n\} &= \mathbb{P}\{X_k < n\mathbb{E}\eta/4 \text{ for some } k \in (n/2, n]\} \\ &\leq \mathbb{P}\{S_k < n\mathbb{E}\eta/4 \text{ for some } k \in (n/2, n]\} \\ &\leq \mathbb{P}\{\inf_{k \leq n} (S_k - k\mathbb{E}\eta) < -n\mathbb{E}\eta/4\}. \end{aligned}$$

Applying now the Kolmogorov inequality, we get  $\mathbb{P}\{\theta_n < n\} = O(1/n)$ . From this estimate and (8.69) we obtain

$$|\mathbb{E} e^{iuX_n}| \leq |\mathbb{E} e^{iu\tilde{X}_n}| + O(1/n), \quad u \in \mathbb{R}. \quad (8.70)$$

For every  $k > n/2$  we have

$$\begin{aligned} \left| \mathbb{E} e^{iu\tilde{X}_k} - \mathbb{E} e^{iu\xi} \mathbb{E} e^{iu\tilde{X}_{k-1}} \right| &= \left| \mathbb{E} e^{iu\tilde{X}_{k-1}} \left( \mathbb{E} \{ e^{iu\xi(\tilde{X}_{k-1})} | \tilde{X}_{k-1} \} - \mathbb{E} e^{iu\xi} \right) \right| \\ &\leq \sup_{x > n\mathbb{E}\eta/4} \sup_{|u| \leq \pi} |\mathbb{E} e^{iu\xi(x)} - \mathbb{E} e^{iu\xi}| =: \delta_n. \end{aligned}$$

It follows from the assumptions of the theorem that  $\delta_n = o(1/n)$ . Consequently, for  $m = \lfloor n/2 \rfloor + 1$  we have

$$\begin{aligned} \left| \mathbb{E} e^{iu\tilde{X}_n} - \left( \mathbb{E} e^{iu\xi} \right)^{n-m} \mathbb{E} e^{iu\tilde{X}_m} \right| &= \left| \sum_{k=m+1}^n \left( \mathbb{E} e^{iu\xi} \right)^{n-k} \left( \mathbb{E} e^{iu\tilde{X}_k} - \mathbb{E} e^{iu\xi} \mathbb{E} e^{iu\tilde{X}_{k-1}} \right) \right| \\ &\leq \delta_n \sum_{j=0}^{n-m-1} \left| e^{iu\xi} \right|^j. \end{aligned}$$

Since  $\mathbb{Z}$  is the minimal lattice for  $\xi$ , there exists a positive constant  $a$  such that  $|e^{iu\xi}| \leq e^{-au^2}$ ,  $u \in [-\pi, \pi]$ . This implies that

$$\left| \mathbb{E} e^{iu\tilde{X}_n} \right| \leq e^{-(n-m)au^2} + \delta_n \sum_{j=0}^{n-m-1} e^{-jau^2} \quad (8.71)$$

Combining (8.70) and (8.71), we obtain

$$\begin{aligned} \int_{|\lambda| \in (A, \pi\sqrt{n})} \left| \mathbb{E} e^{i\lambda \frac{X_n - nu}{\sqrt{n}}} \right| d\lambda &\leq \delta_n \sum_{j=0}^{n-m-1} \int_{|\lambda| \in (A, \pi\sqrt{n})} e^{-a\lambda^2 \frac{j}{n}} d\lambda \\ &\quad + \int_{|\lambda| \in (A, \pi\sqrt{n})} e^{-a\lambda^2 \frac{n-m}{n}} d\lambda + O(1/\sqrt{n}) \\ &= O \left( \frac{1}{\sqrt{n}} + \delta_n \left( 2\pi\sqrt{n} + \sum_{j=1}^{n-m-1} \sqrt{\frac{n}{j}} \right) \right) \\ &\quad + \sqrt{\frac{n}{n-m}} \int_{A\sqrt{\frac{n-m}{n}}}^{\infty} e^{-au^2} du. \end{aligned}$$

Recalling that  $\delta_n = o(1/n)$ , we conclude that

$$\limsup_{n \rightarrow \infty} \int_{|\lambda| \in (A, \pi\sqrt{n})} \left| \mathbb{E} e^{i\lambda \frac{X_n - nu}{\sqrt{n}}} \right| d\lambda \leq \sqrt{2} \int_{A/\sqrt{2}}^{\infty} e^{-au^2} du.$$

Letting now  $A \rightarrow \infty$ , we get the desired result.  $\square$

**Theorem 8.17.** *Assume that all the conditions of Theorem 8.16 hold.*

*If  $\xi$  is non-lattice then, for each  $h > 0$ ,*

$$\sum_{k=0}^n \mathbb{P}\{X_k \in (x, x+h]\} = \frac{h}{\mu} \Phi \left( \frac{n\mu - x}{\sqrt{bx/\mu}} \right) + o(1), \quad x \rightarrow \infty,$$

uniformly in  $n$ .

If  $\mathbb{Z}$  is the minimal lattice for  $\xi$  then

$$\sum_{k=0}^n \mathbb{P}\{X_k = x\} = \frac{1}{\mu} \Phi\left(\frac{n\mu - x}{\sqrt{bx/\mu}}\right) + o(1), \quad x \rightarrow \infty,$$

uniformly in  $n$ .

*Proof.* We consider again the lattice case only. By the local limit theorem, for any positive constants  $A$  and  $B$ ,

$$\begin{aligned} \sum_{k=x/\mu-A\sqrt{x}}^{x/\mu+B\sqrt{x}} \mathbb{P}\{X_k = x\} &= \sum_{k=x/\mu-A\sqrt{x}}^{x/\mu+B\sqrt{x}} \frac{1}{\sqrt{2\pi bk}} e^{-(x-\mu k)^2/2bk} + o(1) \\ &= \frac{1}{\mu} \left( \Phi\left(\frac{\mu^{3/2}B}{\sqrt{b}}\right) - \Phi\left(\frac{\mu^{3/2}A}{\sqrt{b}}\right) \right) + o(1). \end{aligned} \quad (8.72)$$

According to the second part of Theorem 8.15, there exists  $\varepsilon(A) \rightarrow 0$  as  $A \rightarrow \infty$  such that

$$\mathbb{P}\left\{\max_{k \leq x/\mu-A\sqrt{x}} X_k \geq x\right\} \leq \varepsilon(A), \quad x > 0. \quad (8.73)$$

Furthermore, the existence of a minorant with positive mean implies that the conditions of Theorem 8.6 are valid. Therefore, the family  $\sum_{k=0}^{\infty} \mathbb{I}\{X_k = x\}$  is uniform integrable. From this fact and (8.73) we conclude that, as  $A \rightarrow \infty$ ,

$$\sum_{k=0}^{x/\mu-A\sqrt{x}} \mathbb{P}\{X_k = x\} \leq \mathbb{E}\left\{\sum_{k=0}^{\infty} \mathbb{I}\{X_k = x\}; \max_{k \leq x/\mu-A\sqrt{x}} X_k \geq x\right\} \rightarrow 0$$

uniformly in  $x$ . Combining this with (8.72), we get the desired relation.  $\square$

## 8.4 Pre-limiting distributions

In this subsection we shall again always assume that the distribution of  $X_n$  converges towards the stationary distribution  $\pi$  in the total variation metric.

**Theorem 8.18.** *Assume that the conditions of Theorem 8.9 are valid and that the majorant  $\Xi$  satisfies also the condition*

$$\mathbb{E}\Xi^2 e^{\beta\Xi} < \infty. \quad (8.74)$$

Assume also that

$$\mathbb{E}\xi(x) e^{\beta\xi(x)} = \mathbb{E}\xi^{(\beta)} + o(1/\sqrt{x}). \quad (8.75)$$

If the limiting variable  $\xi$  is non-lattice we assume that, for any  $A > 0$ ,

$$\sup_{|\lambda| \leq A} \left| \mathbb{E} e^{(\beta+i\lambda)\xi(x)} - \mathbb{E} e^{(\beta+i\lambda)\xi} \right| = o(1/x). \quad (8.76)$$

If  $\mathbb{Z}$  is the minimal lattice for  $\xi$  we assume that

$$\sup_{|\lambda| \leq \pi} \left| \mathbb{E} e^{(\beta+i\lambda)\xi(x)} - \mathbb{E} e^{(\beta+i\lambda)\xi} \right| = o(1/x). \quad (8.77)$$

Then, uniformly in  $n \geq 1$ ,

$$\frac{\mathbb{P}\{X_n \in (x, x+d]\}}{\pi(x, x+d]} = \Phi\left(\frac{n\mathbb{E}\xi^{(\beta)} - x}{\sqrt{x\mathbb{E}(\xi^{(\beta)})^2/\mathbb{E}\xi^{(\beta)}}}\right) + o(1) \quad \text{as } x \rightarrow \infty,$$

where  $d$  is an arbitrary positive number in the non-lattice case and natural in the lattice case.

*Proof.* Let  $\widehat{X}_n$  be the Markov chain constructed in the proof of Theorem 8.9. There we have shown that the family  $\widehat{\xi}(x)$  possesses a stochastic minorant with positive mean and a stochastic majorant with finite mean. Assumption (8.74) implies that we can take a majorant with finite second moment.

We now turn to the asymptotic behaviour of  $\mathbb{E}\widehat{\xi}(x)$ . As we have shown in the proof of Theorem 8.9,  $\mathbb{E}\widehat{\xi}(x) \rightarrow \mathbb{E}\xi^{(\beta)}$ . But, in order to apply Theorem 8.17, we have to show that

$$\mathbb{E}\widehat{\xi}(x) = \mathbb{E}\xi^{(\beta)} + o(1/\sqrt{x}). \quad (8.78)$$

It follows from (8.47) that

$$\mathbb{E}\widehat{\xi}(x) = \frac{\mathbb{E}\xi(x)U_p(x + \xi(x))}{U_p(x)}(1 + o(1/x)). \quad (8.79)$$

It is immediate from the definition of  $U_p$  that

$$\mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) > s(x)\} \leq U_p(x)\mathbb{E}\{\xi(x)e^{\beta\xi(x)}; \xi(x) > s(x)\}.$$

Due to (8.74), there exists  $s(x) = o(x)$  such that

$$\frac{\mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) > s(x)\}}{U_p(x)} = o(1/s(x)). \quad (8.80)$$

Furthermore, we have an obvious bound

$$\frac{\mathbb{E}\{\xi(x)U_p(x + \xi(x)); \xi(x) < -s(x)\}}{U_p(x)} = o(e^{-\beta s(x)}). \quad (8.81)$$

Uniformly on the set  $\{|\xi(x)| \leq s(x)\}$  we have  $g(x + \xi(x)) - g(x) \sim -p(x)\xi(x)$ . Therefore,

$$\begin{aligned} & \mathbb{E}\{\xi(x)U_p(x + \xi(x)); |\xi(x)| < s(x)\} \\ &= e^{\beta x} \mathbb{E}\{\xi(x)(1 + g(x + \xi(x)))e^{\beta\xi(x)}; |\xi(x)| \leq s(x)\} \\ &= U_p(x) \mathbb{E}\{\xi(x)e^{\beta\xi(x)}; |\xi(x)| \leq s(x)\} - p(x)e^{\beta x} \mathbb{E}\{\xi^2(x)e^{\beta\xi(x)}; |\xi(x)| \leq s(x)\}. \end{aligned}$$

Using again (8.74), we obtain

$$\frac{\mathbb{E}\{\xi(x)U_p(x+\xi(x)); |\xi(x)| < s(x)\}}{U_p(x)} = \mathbb{E}\{\xi(x)e^{\beta\xi(x)}\} + O(p(x) + 1/s(x)).$$

Combining this estimate with (8.80) and (8.81), and choosing  $s(x) \gg \sqrt{x}$ , we conclude that

$$\frac{\mathbb{E}\{\xi(x)U_p(x+\xi(x))\}}{U_p(x)} = \mathbb{E}\{\xi(x)e^{\beta\xi(x)}\} + o(1/\sqrt{x}).$$

(8.78) follows now from the assumption (8.75). The same arguments show that (8.66) and (8.67) follow from (8.76) and (8.77) respectively. Thus,  $\hat{X}_n$  satisfies all the conditions of Theorem 8.17.

Splitting the trajectory of  $X_n$  by the last visit to the compact  $B = [0, \hat{x}]$ , we have

$$\mathbb{P}\{X_n \in dy\} = \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dz\} \frac{U_p(z)}{U_p(y)} \mathbb{E}_z\{e^{-\sum_{k=0}^{j-1} q(\hat{X}_k)}; \hat{X}_j \in dy\}.$$

First of all we note that this representation implies that

$$\mathbb{P}\{X_n \in (x, x+d]\} \leq \frac{1}{U_p(x)} \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dz\} U_p(z) \mathbb{P}_z\{\hat{X}_j > x\}.$$

Recalling that  $\hat{\xi}(x)$  have a square integrable majorant  $\hat{\Xi}$  and applying the Chebyshev inequality, we get

$$\mathbb{P}_z\{\hat{X}_j > x\} = o(x^{-1}) \quad \text{for all } j \leq x/2\mathbb{E}\hat{\Xi}.$$

Consequently,

$$U_p(x) \mathbb{P}\{X_n \in (x, x+d]\} \rightarrow 0$$

uniformly in  $n \leq x/2\mathbb{E}\hat{\Xi}$ . So, it remains to consider the case  $n \geq x/2\mathbb{E}\hat{\Xi}$ . Fix a sequence  $N_x \rightarrow \infty$  such that  $N_x = o(\sqrt{x})$ . Then, by Theorem 8.16,

$$\begin{aligned} & \sum_{j=n-N_x+1}^n \int_B \mathbb{P}\{X_{n-j} \in dz\} \int_x^{x+d} \frac{U_p(z)}{U_p(y)} \mathbb{E}_z\{e^{-\sum_{k=0}^{j-1} q(\hat{X}_k)}; \hat{X}_j \in dy\} \\ & \leq \frac{1}{U_p(x)} \sum_{j=1}^n \int_B \mathbb{P}\{X_{n-j} \in dz\} U_p(z) \mathbb{P}_z\{\hat{X}_j \in (x, x+d]\} \\ & \leq \frac{CN_x}{U_p(x)\sqrt{n}} = o(e^{-\beta x}). \end{aligned} \tag{8.82}$$

Since the distribution of  $X_{n-j}$  converges in total variation to  $\pi$ ,

$$\begin{aligned} & \sum_{j=1}^{n-N_x} \int_B \mathbb{P}\{X_{n-j} \in dz\} \int_x^{x+d} \frac{U_p(z)}{U_p(y)} \mathbb{E}_z\{e^{-\sum_{k=0}^{j-1} q(\hat{X}_k)}; \hat{X}_j \in dy\} \\ & = (1 + o(1)) \sum_{j=1}^{n-N_x} \int_B \pi(dz) \int_x^{x+d} \frac{U_p(z)}{U_p(y)} \mathbb{E}_z\{e^{-\sum_{k=0}^{j-1} q(\hat{X}_k)}; \hat{X}_j \in dy\}. \end{aligned} \tag{8.83}$$

Similar to (8.82),

$$\sum_{j=n-N_x+1}^n \int_B \pi(dz) \int_x^{x+d} \frac{U_p(z)}{U_p(y)} \mathbb{E}_z \{ e^{-\sum_{k=0}^{j-1} q(\hat{X}_k)}; \hat{X}_j \in dy \} = o(e^{-\beta x}).$$

Combining this with (8.82) and (8.83), we obtain

$$\mathbb{P}\{X_n \in (x, x+d]\} = (c + o(1)) \int_x^{x+d} \frac{\hat{H}_n^{(q)}(dy)}{U_p(y)}, \quad (8.84)$$

where

$$\hat{H}_n^{(q)}(dy) = \sum_{j=1}^n \mathbb{E} \{ e^{-\sum_{k=0}^{j-1} q(\hat{X}_k)}; \hat{X}_j \in dy \}$$

Assume that  $\xi$  is lattice. Then

$$\mathbb{P}\{X_n = x\} = (c + o(1)) \frac{\hat{H}_n^{(q)}\{x\}}{U_p(x)}.$$

The claim follows now from Theorem 8.17 and Theorem 8.9. The non-lattice case can be considered in the same way.  $\square$

We can determine the asymptotic behaviour of pre-stationary distribution also in the case when (8.36) is not valid.

**Theorem 8.19.** *Assume that the conditions of Theorem 8.11 are valid. Assume also that*

$$\mathbb{E}\xi(x)e^{\beta(x)\xi(x)} = \mathbb{E}\xi^{(\beta)} + o(1/\sqrt{x}).$$

*If the limiting variable  $\xi$  is non-lattice we assume that, for any  $A > 0$ ,*

$$\sup_{|\lambda| \leq A} \left| \mathbb{E}e^{(\beta(x)+i\lambda)\xi(x)} - \mathbb{E}e^{(\beta+i\lambda)\xi} \right| = o(1/x).$$

*If  $\mathbb{Z}$  is the minimal lattice for  $\xi$  we assume that*

$$\sup_{|\lambda| \leq \pi} \left| \mathbb{E}e^{(\beta(x)+i\lambda)\xi(x)} - \mathbb{E}e^{(\beta+i\lambda)\xi} \right| = o(1/x).$$

*Then, uniformly in  $n \geq 1$ ,*

$$\frac{\mathbb{P}\{X_n \in (x, x+d]\}}{\pi(x, x+1]} = \Phi \left( \frac{n\mathbb{E}\xi^{(\beta)} - x}{\sqrt{x\mathbb{E}(\xi^{(\beta)})^2/\mathbb{E}\xi^{(\beta)}}} \right) + o(1) \quad \text{as } x \rightarrow \infty.$$

The proof of this theorem is identical to that of Theorem 8.18 and we omit it.





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